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ERROR ESTIMATION BASED ON NON-LINEAR ENHANCED ASSUMED STRAIN ELEMENTS

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Abstract. *This paper describes a methodology for error estimation with enhanced assumed strain elements applied to linear solid mechanics problems, finite strain problems and plasticity.*

The relation between the enhanced strain modes and the quality of the finite element solution is analysed for problems of solid mechanics. The analysis is developed in the context of error estimation. Based on the work of (Radovitzky and Ortiz, 1998) for error estimation in strongly non-linear problems, the contribution of the enhanced strain modes is quantified with an energy norm. The methodology proposed for error estimation has the advantages of a) being a local formulation, b) computing the error in an element-by-element way, and c) of having a simple interpretation from a practical point of view.

In the paper firstly the general formulation of the error estimator is described. Following it is particularised for linear and non-linear elasticity and plasticity problems. Representative numerical simulations are presented for 3D non linear elasticity and Von Mises plasticity, with emphasis in the distribution of the local error and the global rate of convergence

1 INTRODUCTION

The finite element method is a common procedure in the cycle of design and verification of engineering structures. However, the value of the finite element solution is not enough. It is necessary to know the quality of the computed results.

The interest of "a posteriori" error estimators lis on their direct applicability to adaptive refinement techniques. The development of these estimators began in the seventies with the pioneering works of (Babuška and Rheinboldt, 1978). From this time till now, several error estimators have been proposed for linear analyses, whose efficiency has been proved in a wide variety of problems.

Nevertheless, in non linear problems developments have not been made until recently, and a number of research lines remain open. Specially remarkable are the developments of (Ortiz and Quigley, 1991) in localisation, (Johnson and Hansbo, 1992) in the context of the elastic-plastic model of Hencky, the error estimator of (Barthold et al., 1998) that is applied to the elastic-plastic models of Hencky and Prandtl-Reuss, etc. Finally it's necessary to remark the recent works of (Radovitzky and Ortiz, 1998) in error estimation for highly non linear problems, including finite deformations in hyperelasticity, viscoplasticity, dynamics, etc.

In this paper a methodology for error estimation is described. The proposed methodology gives a bound of the discretisation error associated to the finite element solution computed with the standard displacement formulation. This error is computed through the enhanced assumed strain (Simó and Rifai, 1990; Simó and Armero, 1993) finite element solution (section 2). For error estimation a variational structure of the boundary value problem is required. This requirement and its influence in local and global error estimation is analysed in section 3. The general expression of the error estimator is described in section 4. The details corresponding to finite elasticity problems and Von Mises plasticity are explained in section 5. Finally, in section 6 some representative numerical simulations are shown, and section 7 describes the conclusions.

2 ENHANCED ASSUMED STRAIN FINITE ELEMENT FORMULATION

The *enhanced assumed strain* finite element formulation (Simó and Rifai, 1990; Simó and Armero, 1993; Simó et al., 1993; Armero and Glaser, 1997) is based on the discrete variational equations obtained from the Hu-Washizu functional (Washizu, 1982). We assume the existence, in each point $\boldsymbol{x} \in \Omega$, of a function of internal energy per unit of volume. This function depends on the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, or on the deformation gradient $\boldsymbol{F}$ if the element is formulated for geometrically non linear problems.

For infinitesimal strains the key ingredient is the additive decomposition of the strain field in a compatible part and an enhanced part:

$$\boldsymbol{\varepsilon} = \underbrace{\nabla^s \boldsymbol{u}}_{\text{compatible}} + \underbrace{\tilde{\boldsymbol{\varepsilon}}}_{\text{enhanced}} ; \tag{1}$$

where $\nabla^s \boldsymbol{u}$ (symmetric component of displacement gradient) is the *"compatible"* part of strain field, and $\tilde{\boldsymbol{\varepsilon}}$ is the enhanced (or incompatible) one. This denomination is motivated for the enhancement of the approximated solution associated with the incompatible part in discrete meshes (for the exact solution the field $\tilde{\boldsymbol{\varepsilon}}$ is null). There are no requirements of inter-element continuity for the enhanced fields $\tilde{\boldsymbol{\varepsilon}}$.

For finite deformation the independent fields are displacements, first Piola-Kirchhoff stress and deformation gradients which are parametrised via the following additive decomposition:

$$\boldsymbol{F} = \underbrace{\boldsymbol{\nabla}_X \boldsymbol{\varphi}}_{\text{compatible}} + \underbrace{\tilde{\boldsymbol{F}}}_{\text{enhanced}} \tag{2}$$

where $\boldsymbol{\nabla}_X$ is the gradient operator and $\boldsymbol{\varphi}$ is the deformation mapping.

3 ERROR ESTIMATION METHODOLOGY

This section describes the general framework for the error estimation methodology. To this end the variational structure of the boundary value problem, the methodology of approximation via finite element method and the requirements for the formulation of local and global error estimators, are explained.

3.1 Variational structure of the boundary value problem

Consider a boundary value problem being $\boldsymbol{u}$ the unknown field. Assume the problem may be expressed via a differential operator $\boldsymbol{A}$:

$$\boldsymbol{A}(\boldsymbol{u}) = \boldsymbol{0} \quad \text{in } \Omega \cup \partial\Omega \tag{3}$$

In the following, expression (3) shall be applied in the context of solid mechanics being $\boldsymbol{u}$ the displacement field. If the problem (3) has variational structure, the Dirichlet form $a(\boldsymbol{u})[\boldsymbol{\eta}, \boldsymbol{\eta}]$ associated to the functional $\Pi_p(\boldsymbol{u})$ is defined as:

$$a(\boldsymbol{u})[\boldsymbol{\eta}, \boldsymbol{\eta}] = \int_\Omega \frac{\partial^2 W(\boldsymbol{\varepsilon}, \boldsymbol{x})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \eta_{i,j} \eta_{k,l} d\Omega \tag{4}$$

being $\boldsymbol{\eta} \in V : \Omega \to \mathbb{R}^n$ admissible variations, V is the space of functions with finite energy and W is the density of internal energy function.

The Dirichlet form $a(\boldsymbol{u})[\boldsymbol{\eta}, \boldsymbol{\eta}]$ is *regular* if the following conditions are verified:

1. $$C_{ijkl} = \frac{\partial^2 W(\boldsymbol{\varepsilon}, \boldsymbol{x})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} < \infty, \ \forall \boldsymbol{x} \in \Omega \Leftrightarrow C_{ijkl} \in L^\infty(\Omega, \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \tag{5}$$
2. $$a(\boldsymbol{u})[\boldsymbol{\eta}, \boldsymbol{\eta}] > C \|\boldsymbol{\eta}\|_{1,2}^2 \quad C \in \mathbb{R}^+ \text{ (coercivity condition)} \tag{6}$$

where L^∞ is the Lebesgue space of order infinity and $\|\cdot\|_{1,2}$ is the Sobolev norm with degree 1 and order 2.

If the Dirichlet form (4) verifies the regularity hypotheses (5, 6), then the following conditions are asserted:

i) Π_p is convex

ii) Π_p has a unique relative minimum; hence, the solution $\boldsymbol{u}$ of the boundary value problem verifies:

$$\Pi_p(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in V} \Pi_p(\boldsymbol{v}) \tag{7}$$

3.2 Methodology of approximation with compatible elements

For infinitesimal elasticity, the Euler-Lagrange equations of the variational principle associated to the functional $\Pi_p(\boldsymbol{u})$ are:

$$a(\boldsymbol{u})[\boldsymbol{u}, \boldsymbol{\eta}] - < \boldsymbol{b}, \boldsymbol{\eta} >= 0 \quad \forall \boldsymbol{\eta} \in V \tag{8}$$

being $< \boldsymbol{b}, \boldsymbol{\eta} >$ the inner product:

$$< \boldsymbol{b}, \boldsymbol{\eta} >= \int_\Omega \boldsymbol{b} \cdot \boldsymbol{\eta} d\Omega$$

Assume a finite dimension subspace $V_h \subset V | \dim(V_h) < \infty i$ and $V_h \to V$ when $h \to 0$. Formulating (8) in V_h results in:

$$a(\boldsymbol{u})[\boldsymbol{u}_h, \boldsymbol{\eta}_h] - < \boldsymbol{b}, \boldsymbol{\eta}_h >= 0 \quad \forall \boldsymbol{\eta}_h \in V_h \tag{9}$$

Since $V_h \subset V$, applying (8) to the elements of V_h it is verified that:

$$a(\boldsymbol{u})[\boldsymbol{u}, \boldsymbol{\eta}_h] - < \boldsymbol{b}, \boldsymbol{\eta}_h >= 0 \quad \forall \boldsymbol{\eta}_h \in V_h \tag{10}$$

Subtracting (9) and (10), if the problem is linear, the following result is obtained:

$$a(\boldsymbol{u})[\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\eta}_h] = 0 \quad \forall \boldsymbol{\eta}_h \in V_h \tag{11}$$

Equation (11) establishes that the finite element solution minimises the value of $\|\boldsymbol{u} - \boldsymbol{u}_h\|_E$. This property is referred to as the *optimal approximation property* of the finite element method:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_E = \inf_{\boldsymbol{v_h}\in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_E \tag{12}$$

3.3 Local error estimation

In general, the finite element solution is obtained in the discrete domain Ω^h, which is constructed via the discretisation of the domain Ω, using n_{el} elements Ω^e, such that:

$$\begin{aligned} \bigcup_{e=1}^{n_{el}} \Omega^e &= \Omega_h \\ \Omega_i^e \cap \Omega_j^e &= \emptyset \quad \forall i \neq j \end{aligned}$$

Let Ω^e be an element in $\mathbb{R}^n$ with positive jacobian determinant, and let $\mathbb{P}_p(\Omega^e)$ be the set of polynomials over Ω^e with degree lower or equal than p. Let $\boldsymbol{u}^e \in H^1(\Omega^e, \mathbb{R}^n)$ be the exact displacement field in the element e, and let $\boldsymbol{u}_h^e(\boldsymbol{x}) = \sum_{a=1}^{n_{node}} \boldsymbol{u}_a N_a(\boldsymbol{x}) \in \mathbb{P}_p(\Omega^e)$ be the "finite dimension interpolant polynomial" of the exact solution $\boldsymbol{u}^e$, where n_{node} is the number of nodes of the element e.

The local error function in the element e is defined as the difference between the exact displacement field and the displacement field computed via the finite element method:

$$\boldsymbol{E}^e(\boldsymbol{x}) = \boldsymbol{u}^e(\boldsymbol{x}) - \boldsymbol{u}_h^e(\boldsymbol{x})$$

The problem to solve with a local error estimator is to obtain an upper bound of the local error function, which may be expressed in the following way:

$$\|\boldsymbol{u}^e - \boldsymbol{u}_h^e\| \leq C(h^e)^\alpha |\boldsymbol{u}^e| \tag{13}$$

where:

C	:	real positive constant
h^e	:	diameter of the circunference circumscribed around Ω^e
$\lvert\boldsymbol{u}^e\rvert$	:	seminorm of $\boldsymbol{u}^e$
α	:	rate of convergence

The definition of the semi-norm used in (13) is independent of the definition of the error norm established. The equality (13) is verified if the *optimal approximation property* (12) and the regularity conditions expressed in (5; 6) are satisfied.

3.4 Global error estimation

From the expression of the interpolation functions of $\boldsymbol{u}^e(\boldsymbol{x})$,

$$\boldsymbol{u}_h^e(\boldsymbol{x}) = \sum_{a=1}^{n_{node}} \boldsymbol{u}_a N_a(\boldsymbol{x})$$

the *Global interpolant polynomial* $\boldsymbol{u}_h(\boldsymbol{x})$ is defined as:

$$\boldsymbol{u}_h(\boldsymbol{x}) = \sum_{e=1}^{n_{el}} \boldsymbol{u}_h^e(\boldsymbol{x})$$

If the shape functions are conforming, the following is satisfied: $\boldsymbol{u}_h^e(\boldsymbol{x}) \in H^1(\Omega^e, \mathbb{R}^n) \Rightarrow \boldsymbol{u}_h(\boldsymbol{x}) \in H^1(\Omega, \mathbb{R}^n)$, where H^1 is the Sobolev space of order 1.

In order to write an upper bound of the global error function: $\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_h(\boldsymbol{x})$, the seminorm of $\boldsymbol{E}(\boldsymbol{x})$ used in (13) is expressed as the summation of the contributions of each element. Using the energy norm, this results in (Ciarlet, 1978):

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,2} \leq \sum_{e=1}^{n_{el}} C \frac{(h^e)^2}{\rho^e} |\boldsymbol{u}^e|_{2,2} \tag{14}$$

The expression (14) shows that the upper bound of the global error may be expressed as the sum of the local error bounds computed in each element. Besides, if regularity conditions (5,6) hold and taking into account the inequality of Poincaré, the semi norm $|\cdot|_{1,2}$ can be replaced by the energy norm in (14), resulting:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_E \leq C \sum_{e=1}^{n_{el}} \frac{(h^e)^2}{\rho^e} |\boldsymbol{u}^e|_{2,2} \tag{15}$$

4 ERROR ESTIMATOR PROPOSED

From a practical point of view, equation (15) is not convenient because the error is expressed in terms of the unknown field $\boldsymbol{u}^e$. Besides it is not possible to substitute this field by its approximate solution $\boldsymbol{u}_h^e$, as it is a polynomial of degree k and the seminorm used is of order $k+1$ ($D^\alpha \boldsymbol{u}_h^e = 0$ for $|\alpha| = k+1$)

Error estimation techniques are based on the substitution of $\boldsymbol{u}^e$ by another field, in such manner that the estimated error must be a realistic measure. The methodology for performing this substitution leads to different error estimators.

The error estimator for the solution $\boldsymbol{u}_h$ (obtained with elements formulated in displacements), which employed in this paper and more fully described in (Gabaldón, 1999), is based on the solution $\boldsymbol{u}_{\mathrm{enh}}$ obtained with the enhanced assumed strain elements described in section 2. From the triangular inequality:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_E \leq \|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{enh}}\|_E + \|\boldsymbol{u}_{\mathrm{enh}} - \boldsymbol{u}_h\|_E \tag{16}$$

It is assumed that the rates of convergence are:

$$\|\boldsymbol{u} - \boldsymbol{u}_{\mathrm{enh}}\|_E = o(h^m) \tag{17}$$

$$\|\boldsymbol{u}_{\mathrm{enh}} - \boldsymbol{u}_h\|_E = o(h^p) \tag{18}$$

also, at least in the asymptotic regime, the following hypothesis holds:

$$m > p \tag{19}$$

In these conditions, at least for $h \to 0$, in the right hand side of equation (16) the first term is negligible if it is compared to the second one. Therefore it is possible to establish that:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_E \leq C \|\boldsymbol{u}_{\mathrm{enh}} - \boldsymbol{u}_h\|_E \quad C \in \mathbb{R}^+ \tag{20}$$

The hypotheses (17;18;19) may be re-interpreted in the following terms: The solutions $\boldsymbol{u}_{\mathrm{enh}}$ and $\boldsymbol{u}_h$ converge to the exact solution in such manner that

1. $\|\boldsymbol{u}_{\text{enh}} - \boldsymbol{u}_h\|_E$ decreases with the refinement of the mesh;

2. The solution obtained with enhanced elements is a better approximation to the exact solution than the solution of standard elements to the enhanced ones.

The expression of the local estimator proposed is:

$$(E^e)^2 = \|\boldsymbol{u}^e_{\text{enh}} - \boldsymbol{u}^e_h\|_E \tag{21}$$

In accordance to the previous section, the global error may be obtained as the sum of the local errors:

$$E^2 = \sum_{i=1}^{n_{el}} (E^i)^2 \tag{22}$$

The discretisation error associated to the standard elements is quantified via the internal energy associated to the incompatible modes computed with enhanced elements.

Each component in the sum (22) is local, and therefore the proposed estimator has the important advantage that is computed element by element, without global smoothing techniques nor sub-domain solutions.

5 ENERGY CONTRIBUTION OF THE INCOMPATIBLE MODES

In this section the application of (21) to error estimation problems is explained. Finite elasticity problems with hyperelastic constitutive models and small strain problems with Von Mises plasticity are considered.

5.1 Finite elasticity

Here the unknown field is the deformation mapping $\boldsymbol{\varphi} : \Omega \to \Omega_t$, where Ω is the reference configuration and Ω_t is the deformed configuration at time t. The formulation is similar to what has been already developed in section 3.1, but replacing the displacement field $\boldsymbol{u}$ for the deformation $\boldsymbol{\varphi}$, and the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ for the deformation gradient $\boldsymbol{F}$.

With respect to the approximation methodology via standard elements described in 3.2, subtracting (9; 10) the following result is obtained:

$$a(\boldsymbol{\varphi})[\boldsymbol{\varphi}, \boldsymbol{\eta}_h] - a(\boldsymbol{\varphi}_h)[\boldsymbol{\varphi}_h, \boldsymbol{\eta}_h] = 0 \quad \forall \boldsymbol{\eta}_h \in V_h \tag{23}$$

This is different to (11), as the Dirichlet form $a(\boldsymbol{\varphi})[\cdot, \cdot]$ is *non linear* in finite elasticity. Nevertheless, for the asymptotic regime ($h \to 0$), the finite element solution $\boldsymbol{\varphi}_h$ is approximately equal to the exact solution, and then equation (23) may be linearised resulting in:

$$a(\boldsymbol{\varphi})[\boldsymbol{\varphi} - \boldsymbol{\varphi}_h, \boldsymbol{\eta}_h] = 0 \quad \forall \boldsymbol{\eta}_h \in V_h, \quad h \to 0 \tag{24}$$

This condition establishes the *optimal approximation property* of the finite element method for finite elasticity, in the asymptotic regime:

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_E = \inf_{\boldsymbol{v_h} \in V_h} \|\boldsymbol{\varphi} - \boldsymbol{v}_h\|_E \tag{25}$$

The expression of the local error estimator proposed in (21) results:

$$(E^e)^2 = \|\boldsymbol{\varphi}^e_{\text{enh}} - \boldsymbol{\varphi}^e_h\|_E \tag{26}$$

assuming the hypotheses (17;18;19) hold.

For the numerical implementation, the value of (26) is computed in the reference configuration. Then, the expression of the energy norm is (Gabaldón, 1999):

$$\|\boldsymbol{\varphi}\|^2_E = a(\boldsymbol{\varphi})[\boldsymbol{\varphi}, \boldsymbol{\varphi}] = \int_{\Omega_0} \boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\varphi} \cdot \mathbf{A} \boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\varphi} \, d\Omega \tag{27}$$

where $\mathbf{A}$ is the tangent tensor of constitutive moduli:

$$\mathbf{A} = \frac{\partial^2 W(\boldsymbol{X}, \boldsymbol{F})}{\partial \boldsymbol{F} \partial \boldsymbol{F}} = \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}} \tag{28}$$

Simple calculations provide the expression of the error estimator that has been implemented (Gabaldón, 1999):

$$(E^e)^2 = \frac{1}{2} \int_{\Omega^e} \widetilde{\boldsymbol{F}} \cdot \mathbf{A} \widetilde{\boldsymbol{F}} \, d\Omega \tag{29}$$

where $\widetilde{\boldsymbol{F}}$ is the enhanced part of the deformation gradient (Armero and Glaser, 1997).

Computing the global error via (29) extended over all the domain Ω, it can be expressed as the sum of the local errors:

$$E^2 = \sum_{i=1}^{n_{el}} (E^i)^2 \tag{30}$$

5.2 Plasticity

The methodology for error estimation described in section 3 assumes a variational structure of the boundary value problem. In plasticity, this variational structure may be obtained at an incremental level via the variational integration of the plasticity equations (Ortiz and Stainier, 1998). The variational integration postulates the existence of an incremental energy function per unit volume $W_{t+\Delta t}$, such that

$$\boldsymbol{\sigma}_{t+\Delta t} = \frac{\partial W_{t+\Delta t}}{\partial \boldsymbol{\varepsilon}^e_{t+\Delta t}} \tag{31}$$

In infinitesimal J_2 plasticity with isotropic hardening, the functional dependence of $W_{t+\Delta t}$ is on elastic strain and effective plastic strain ξ. The expression of the incremental potential function is:

$$W_{t+\Delta t}(\boldsymbol{\varepsilon}^e_{t+\Delta t}, \xi_{t+\Delta t}, \boldsymbol{\varepsilon}^e_t, \xi_t) = \min_{\xi_{t+\Delta t}} \left(\Psi_{t+\Delta t}(\boldsymbol{\varepsilon}^e_{t+\Delta t}, \xi_{t+\Delta t}) - \Psi_t(\boldsymbol{\varepsilon}^e_t, \xi_t) \right) \tag{32}$$

where $\Psi(\boldsymbol{\varepsilon}^e, \xi)$ is the free energy function. The minimum requirement in the right-hand side of (32) is equivalent to the condition:

$$\frac{\partial \Psi_{t+\Delta t}(\boldsymbol{\varepsilon}^e_{t+\Delta t}, \xi_{t+\Delta t})}{\partial \xi_{t+\Delta t}} = 0 \tag{33}$$

Assuming that the elastic response is independent of the phenomena associated to unrecoverable distortions of the crystalline lattice, the free energy function may be expressed via the additive decomposition in an elastic part and a plastic part. Besides, if the additive decomposition of the infinitesimal strain tensor is assumed:

$$\Psi(\boldsymbol{\varepsilon}^e, \xi) = \Psi^e(\boldsymbol{\varepsilon}^e) + \Psi^p(\xi); \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p, \tag{34}$$

the incremental potential $W_{t+\Delta t}$ can be written as:

$$W_{t+\Delta t} = \min_{\xi_{t+\Delta t}} \left(\Psi^e_{t+\Delta t}(\boldsymbol{\varepsilon}_{t+\Delta t} - \boldsymbol{\varepsilon}^p_{t+\Delta t}) + \Psi^p_{t+\Delta t}(\xi_{t+\Delta t}) - \Psi^e_t(\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}^p_t) + \Psi^p_t(\xi_t) \right) \tag{35}$$

The optimisation condition (33) applied to (35), leads to the following expression (Gabaldón, 1999):

$$\left(J_{2,_{t+\Delta t}}\right)^2 = \frac{2}{3} \frac{\partial \Psi^p}{\partial \xi_{t+\Delta t}} \tag{36}$$

where J_2 is the second invariant of the deviatoric part of the stress tensor.

In this situation the boundary value problem is described via an incremental differential operator,

$$\boldsymbol{A}_{t+\Delta t} = \boldsymbol{0} \tag{37}$$

The corresponding Dirichlet form is

$$a(\boldsymbol{u}_{t+\Delta t})[\boldsymbol{\eta}, \boldsymbol{\eta}] = \int_\Omega \frac{\partial^2 W_{t+\Delta t}(\boldsymbol{\nabla}^{\mathrm{s}} \boldsymbol{u}, \cdot)}{\partial \boldsymbol{\varepsilon}_{t+\Delta t} \partial \boldsymbol{\varepsilon}_{t+\Delta t}} \cdot \boldsymbol{\nabla}^{\mathrm{s}} \boldsymbol{\eta} \cdot \boldsymbol{\nabla}^{\mathrm{s}} \boldsymbol{\eta} \, d\Omega \tag{38}$$

If the Dirichlet form (38) verifies (5, 6) then it is regular and is applicable the error estimation methodology described in previous sections.

The local error estimator for this kind of problems is:

$$(E^e_{\Delta t})^2 = \| \boldsymbol{u}^e_{\mathrm{enh}_{t+\Delta t}} - \boldsymbol{u}^e_{h_{t+\Delta t}} \|_E \tag{39}$$

The error bound proposed in (39) is an incremental bound. In order to evaluate the discretisation error along the load path, it is necessary to determine the integral of $E^e_{\Delta t}$ over the time:

$$E^e_{t+\Delta t} = \int_0^{t+\Delta t} E^e_{\Delta t} dt \tag{40}$$

Using the incremental function $W_{t+\Delta t}$, the error estimator is interpreted as the contribution of the incompatible modes of the free energy function:

$$(E^e_{\Delta t})^2 = \int_{\Omega_e} W_{t+\Delta t}\left(\boldsymbol{\varepsilon}^e_{t+\Delta t} - \boldsymbol{\varepsilon}^e_{t+\Delta t}(\boldsymbol{u}), \xi_{t+\Delta t} - \xi_{t+\Delta t}(\boldsymbol{u}), \boldsymbol{\varepsilon}^e_t - \boldsymbol{\varepsilon}^e_t(\boldsymbol{u}), \xi_t - \xi_t(\boldsymbol{u})\right) d\Omega \tag{41}$$

The energy density in (41) can be decomposed in an additive way with the contributions of the elastic and plastic part of the of the free energy, resulting in:

$$\begin{aligned}(E^e_{\Delta t})^2 \; = \; & \int_{\Omega_e} W^e_{t+\Delta t}\left(\boldsymbol{\varepsilon}^e_{t+\Delta t} - \boldsymbol{\varepsilon}^e_{t+\Delta t}(\boldsymbol{u}), \boldsymbol{\varepsilon}^e_t - \boldsymbol{\varepsilon}^e_t(\boldsymbol{u})\right) d\Omega + \\ & \int_{\Omega_e} W^p_{t+\Delta t}\left(\xi_{t+\Delta t} - \xi_{t+\Delta t}(\boldsymbol{u}), \xi_t - \xi_t(\boldsymbol{u})\right) d\Omega \end{aligned} \tag{42}$$

The global discretisation error is obtained extending the integral in (41) to the complete domain Ω. Then, the global error is computed via the summation of the local errors:

$$E^2_{\Delta t} = \sum_{i=1}^{n_{el}} (E^i_{\Delta t})^2 \tag{43}$$

6 NUMERICAL SIMULATIONS

6.1 3-D Finite elasticity. Scordelis-Lo roof

This example is a finite deformation version of the classical Scordelis-Lo test for shell elements (Scordelis and Lo, 1969). On an 80° cylindrical sector roof a volumetric load is acting in Z direction. The roof is situated on two diaphragms. Figure 1 shows the problem definition taking into account the symmetry conditions. The hyperelastic constitutive model has the following energy function:

$$W(\boldsymbol{C}) = \frac{1}{2}\lambda(\log J)^2 - \mu\log(J) + \frac{1}{2}\mu(\mathrm{trace}(\boldsymbol{C}) - 3) \tag{44}$$

with $\boldsymbol{C}$ the right Cauchy tensor, J the determinant of the deformation gradient and (λ, μ) the Lamé parameters. The numerical values adopted are: $\lambda = 0$, $\mu = 2.16\cdot 10^8$, $L = 50$, $R = 25$, $t = 0.25$ and $b = 360.0$.

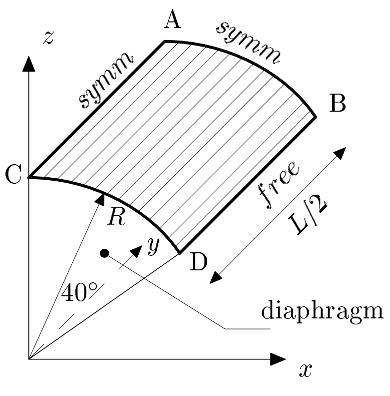

Figure 1: *Scordelis-Lo roof. Definition of the problem.*

The results obtained with enhanced assumed strain elements are very adequate for error estimation because they are similar to those obtained with shell elements, as it's shown in table 1. The results for shell elements referenced in this table have been obtained with the four node thick shell element S4 from a commercial program element library (ABAQUS, 1998). Cancelling the finite deformation terms, the vertical displacement of point B computed with 8×1 enhanced solid elements is $v_B = 0.30070$, very similar to the result reported in the literature.

N^{o} of elements	v_B (solid elements)	v_B (shell elements)
$4 \times 4 \times 1$	0.1587	0.2674
$8 \times 8 \times 1$	0.2561	0.2631
$16 \times 16 \times 1$	0.2670	0.2641
$32 \times 32 \times 1$	0.2681	0.2648
$64 \times 64 \times 1$	0.2684	0.2652

Table 1: *Results obtained with 3D enhanced assumed finite strain elements and thick shell elements for Scordelis-Lo test*

For error estimation the exact solution adopted here is the obtained one with a 25025 D.O.F. mesh, solved with enhanced assumed strain elements. The error estimation is analysed refining over the surface of the roof with only one element along the thickness, as well as refining through the thickness. The meshes considered for the first case and their sizes are shown in figure 2. The dimension of the meshes used for refinement along the thickness are $16 \times 16 \times 1$, $16 \times 16 \times 2$, $16 \times 16 \times 3$, $16 \times 16 \times 4$, $16 \times 16 \times 5$ and $16 \times 16 \times 10$. Figures 3 and 4 show the evolution of the internal energy for the two kinds of refinements, in terms of the number of degrees of freedom.

Figures 5 and 6 show the evolution of the global error estimated in terms of the number of degrees of freedom. The theoretical error is computed as the difference between the reference energy and the energy computed with standard elements. It is remarkable that the refinement along the thickness does not improve the results, neither in energy nor in displacement, and the error estimation reflects adecuately this fact.

Finally, figure 7 shows the error contours obtained with two of the considered meshes.

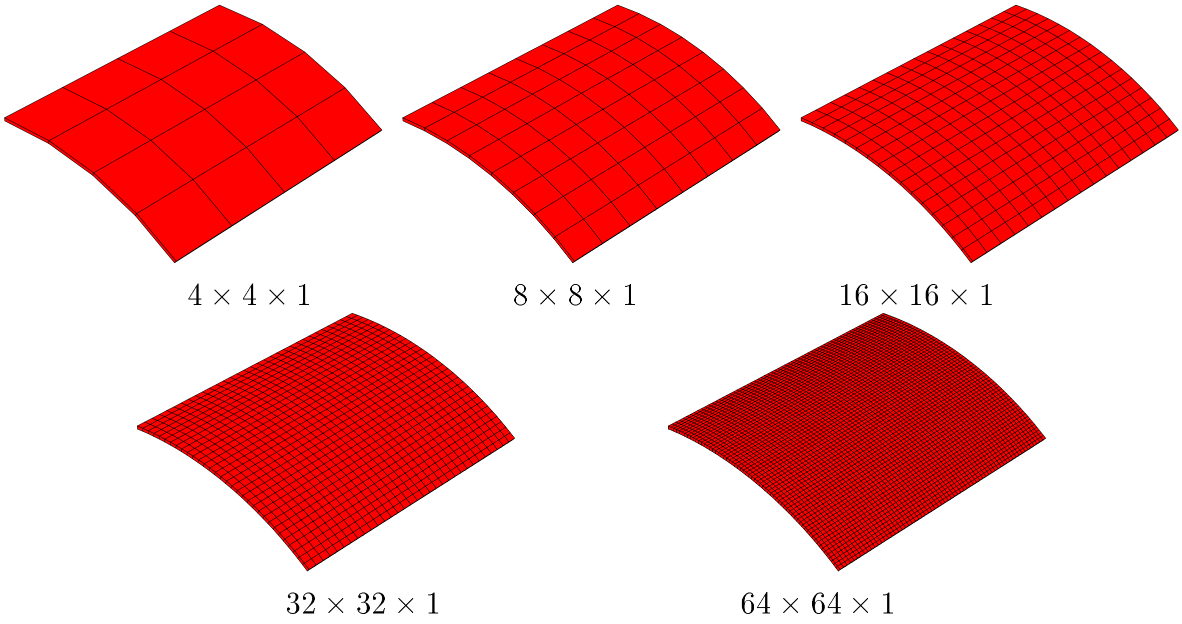

Figure 2: *Refinement over the surface of the roof for error estimation*

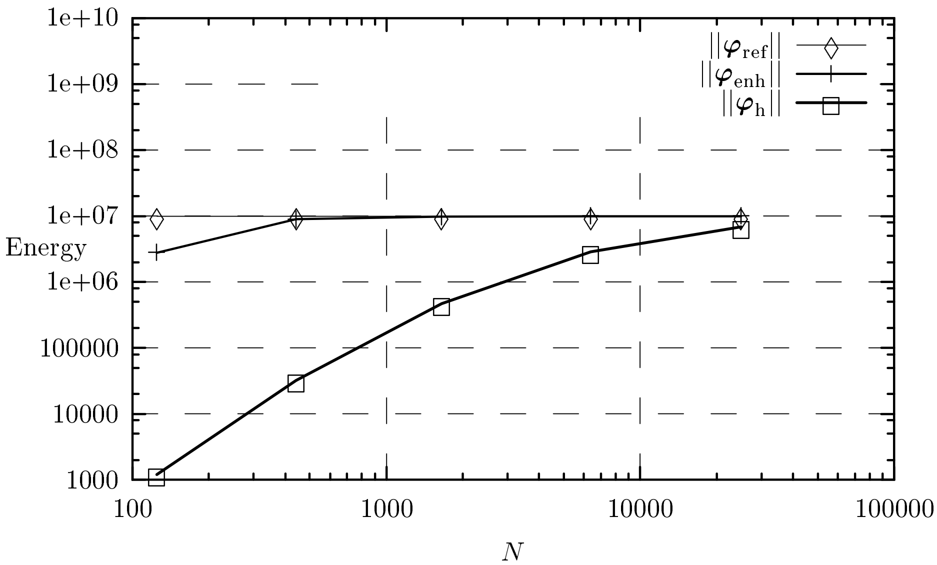

Figure 3: *Variation of internal energy with D.O.F. for refinement over the surface*

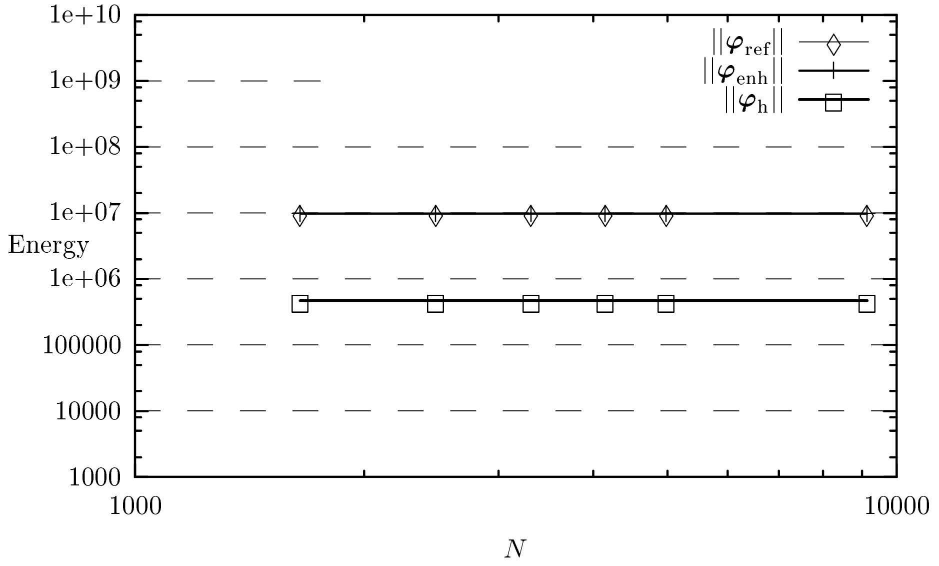

Figure 4: *Variation of internal energy with D.O.F. for refinement along thickness*

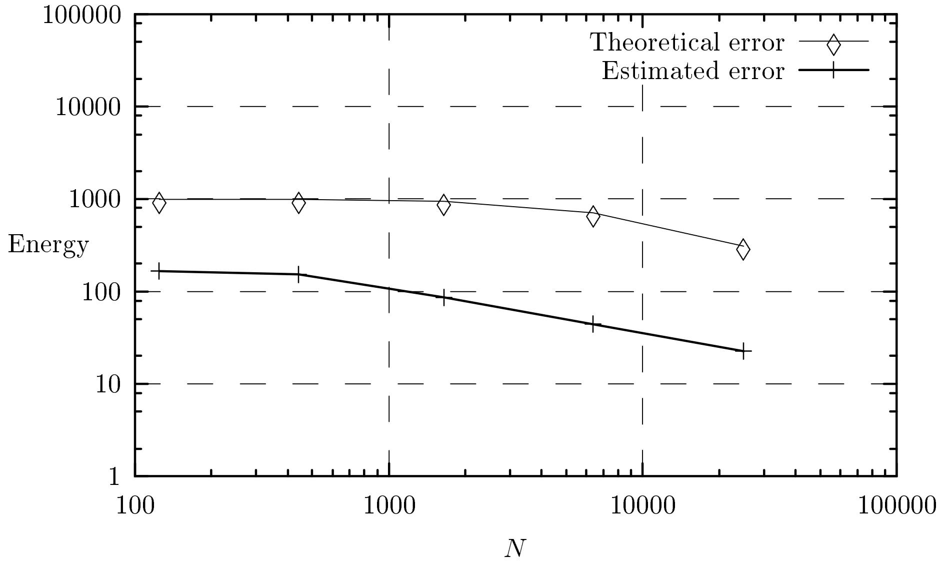

Figure 5: *Evolution of global error in terms of D.O.F. for refinement over the surface*

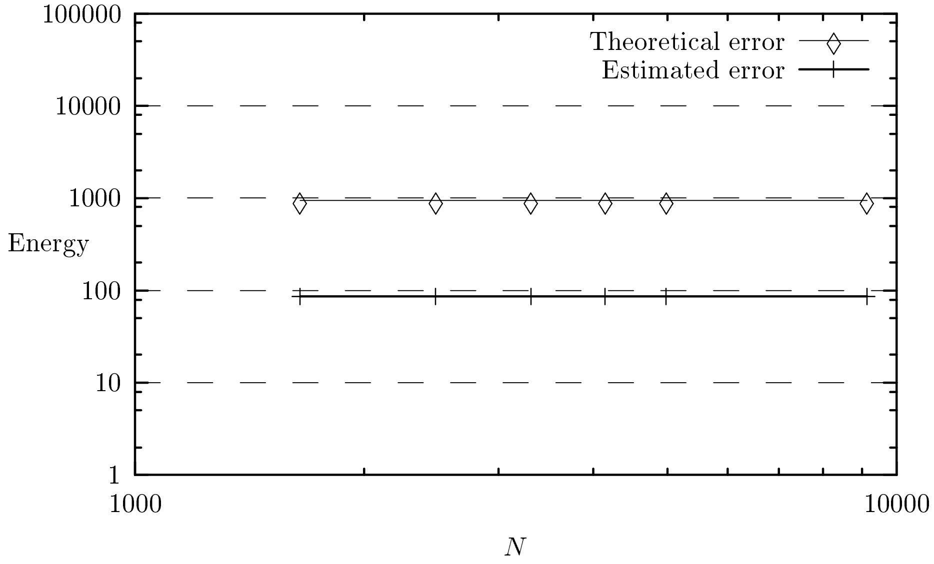

Figure 6: *Evolution of global error in terms of D.O.F. for refinement along thickness*

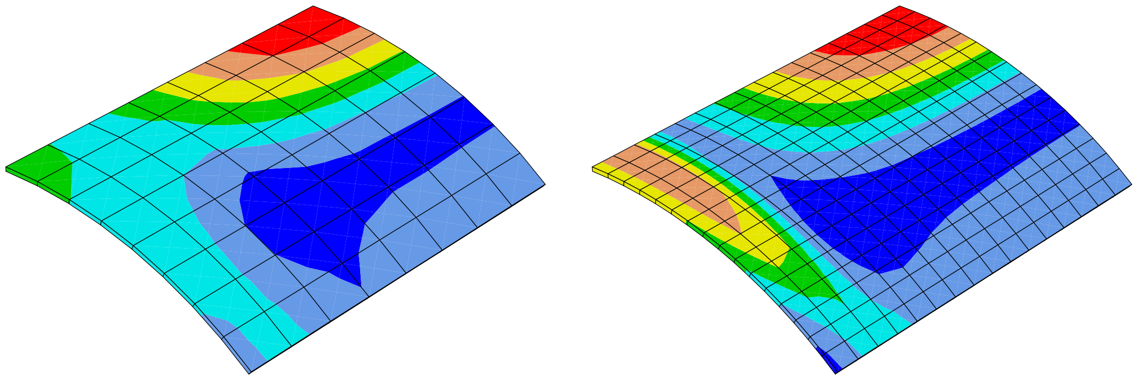

Figure 7: *Contours of iso-error computed for the* $8 \times 8 \times 1$ *and* $16 \times 16 \times 1$ *meshes*

6.2 Plasticity. Square plate with circular hole

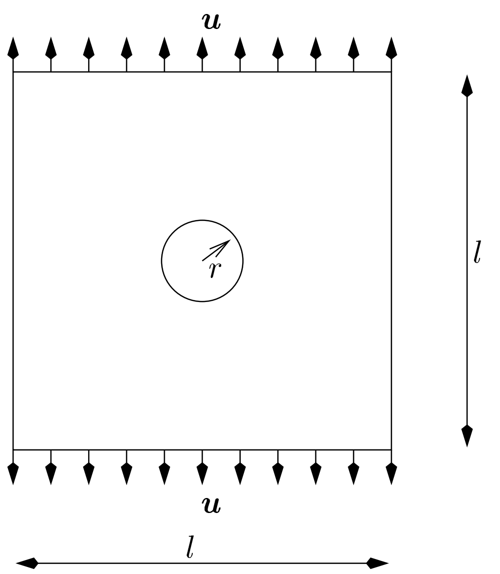

Figure 8: Square plate with circular hole. Geometry and boundary conditions

This example simulates the tensile test of a square plate with a circular hole in plane strain conditions. There are two singular points in the diameter normal to the direction of the imposed displacements. The dimensions and boundary conditions are shown in figure 8, where $l = 50$ and $r = 10$, and $u = 0.32$ is the final value of the imposed displacement. This example has been used as benchmark test for error estimation in a research project of the University of Hannover (Stein et al., 1997), and a summary of the results obtained has been published in (Barthold et al., 1998). The constitutive behaviour is modelled with an elastic-plastic Von Mises model with the following saturation law for hardening:

$$\sigma_Y = \sigma_0 + (\sigma_\infty - \sigma_0)(1 - e^{-\beta\xi}) + H\xi \qquad (45)$$

The numerical values adopted are: $\sigma_0 = 450.0$, $\sigma_\infty = 750.0$, $\beta = 16.93$, $H = 129.0$. The elastic behaviour is defined with the moduli $E = 206900.0$ and $\nu = 0.29$ For the error estimation four meshes with 37, 407, 1583 and 3119 degrees of freedom are considered. Figure 9 shows the curves of internal energy obtained with enhanced elements and the global error estimated at the end of the computation, for the degrees of freedom considered. The values of the internal energy are not dependent on the degrees of freedom because the elastic energy decreases and the irreversible energy increases with the degrees of freedom. The values of the error estimator obtained predict a low order of convergence: the 1/8 slope plotted in double logarithmic scale is well adjusted to the rate of convergence obtained in the computations. Figure 10 shows the local error contours at the end of the process. These values are obtained adding along the load path the incremental values obtained via (41). The maximum values of contours are reached in the singular point, and as the mesh is refined the contours are localised in a 45° band.

Finally the evolution of the local error near the singular point is studied. The error is additively decomposed into an elastic and a plastic part, in accordance to (42). The two components are shown in figure 11. The conclusions obtained from these curves are:

1. The elastic and plastic parts of the local error decrease with mesh refinement, although the plastic part is activated earlier in the refined meshes.

2. The two components increase along the load process

3. The order of magnitude of the two components is similar.

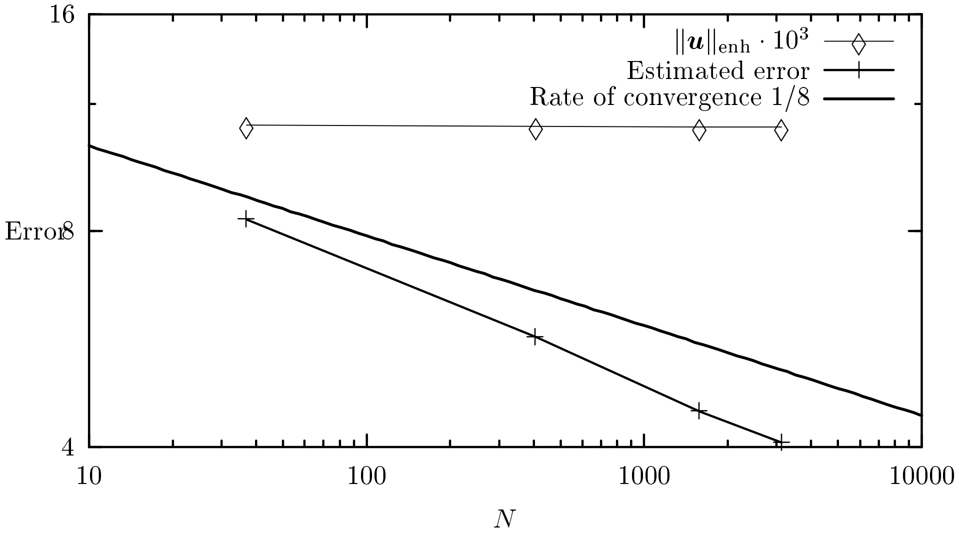

Figure 9: *Square plate with circular hole. Global error estimated*

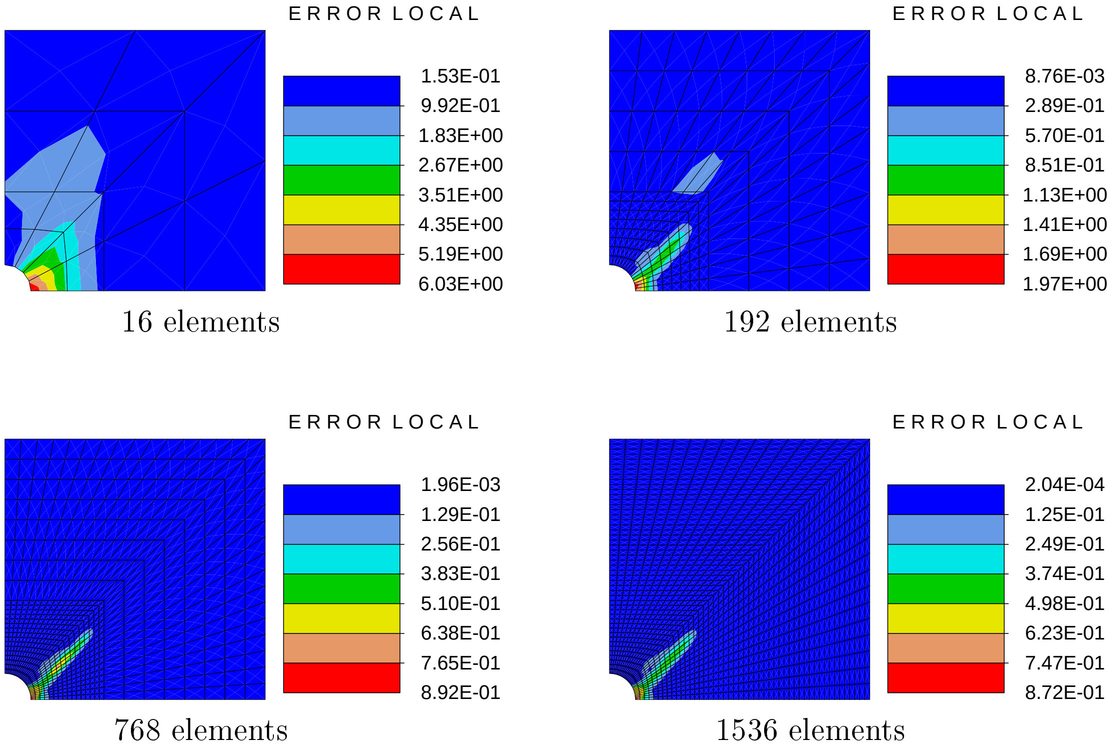

Figure 10: *Square plate with circular hole. Contours of local error*

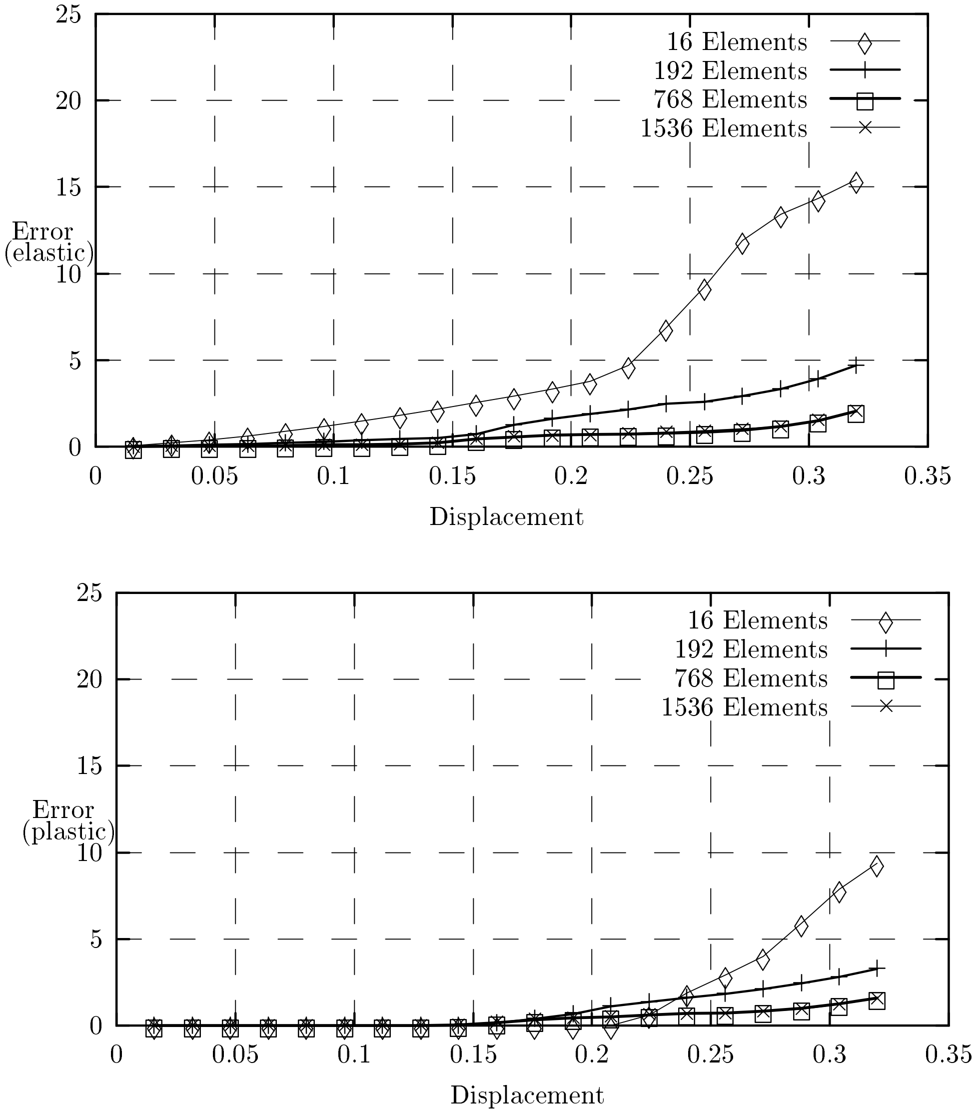

Figure 11: *Square plate with a hole. Evolution of the plastic part of the local error near the singular point.*

7 CONCLUSIONS

In this paper a methodology for error estimation for non linear problems has been described. The methodology is based on the energy contribution of incompatible modes, and has been applied to non-linear finite hyperelasticity and Von-Mises plasticity problems.

The error estimator proposed, based on recent works (Radovitzky and Ortiz, 1998), establishes a measure of the discretisation error obtained with standard elements from the solution computed with enhanced assumed strain elements. It is formulated in a local manner and therefore has a low computational cost.

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