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Contribution of incompatible modes in elastic-plastic and localisation problems

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Acknowledgment: C.García Garino

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- Enhanced assumed strain elements formulation for finite deformation problems
- Constitutive model
- Localisation criteria
- Elastic-plastic unconstrained single element test
- Localisation of a single element
- Localisation of a specimen in tension
- Conclusions

E.A.S. FINITE DEFORMATION

- Key idea (Simo & Armero, 1992):

$$\boldsymbol{F} = \nabla_X \boldsymbol{\varphi} + \tilde{\boldsymbol{F}}$$

where:

$$\boldsymbol{\varphi} : \boldsymbol{X} \mapsto \boldsymbol{x}$$

- Enhanced deformation fields:
  - Q1/E4 element (Simo & Armero, 1992)

$$\mathbb{F} = \Gamma_1 \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} + \Gamma_2 \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} + \Gamma_3 \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix} + \Gamma_4 \begin{pmatrix} 0 & 0 \\ 0 & \eta \end{pmatrix}$$

  - Q1/ES4 element (Armero, 1996)

$$\mathbb{F} = \Gamma_1 \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} + \Gamma_2 \begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix} + \Gamma_3 \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} + \Gamma_4 \begin{pmatrix} 0 & 0 \\ 0 & \eta \end{pmatrix}$$

  - Q1/ET4 element (Armero, 1996)

$$\mathbb{F} = \Gamma_1 \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} + \Gamma_2 \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} + \Gamma_3 \begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix} + \Gamma_4 \begin{pmatrix} 0 & 0 \\ 0 & \eta \end{pmatrix}$$

- Enhanced deformation gradient construction (Armero & Glaser 1996)

$$\tilde{\mathbb{F}} = \frac{j_0}{j} \boldsymbol{J}_0^{-T} \mathbb{F} \boldsymbol{J}_0^{-1} \qquad \tilde{\boldsymbol{F}} = \boldsymbol{F}_0 \tilde{\mathbb{F}}$$

where:

$$\begin{aligned} \boldsymbol{J} &= \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\xi}} \qquad \boldsymbol{J}_0 = \left. \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\xi}} \right|_{\boldsymbol{\xi}=\boldsymbol{0}} \\ j &= \det(\boldsymbol{J}) \qquad j_0 = \det(\boldsymbol{J}_0) \end{aligned}$$

CONSTITUTIVE MODEL

- J_2 plasticity is considered, using the multiplicative decomposition of the deformation gradient

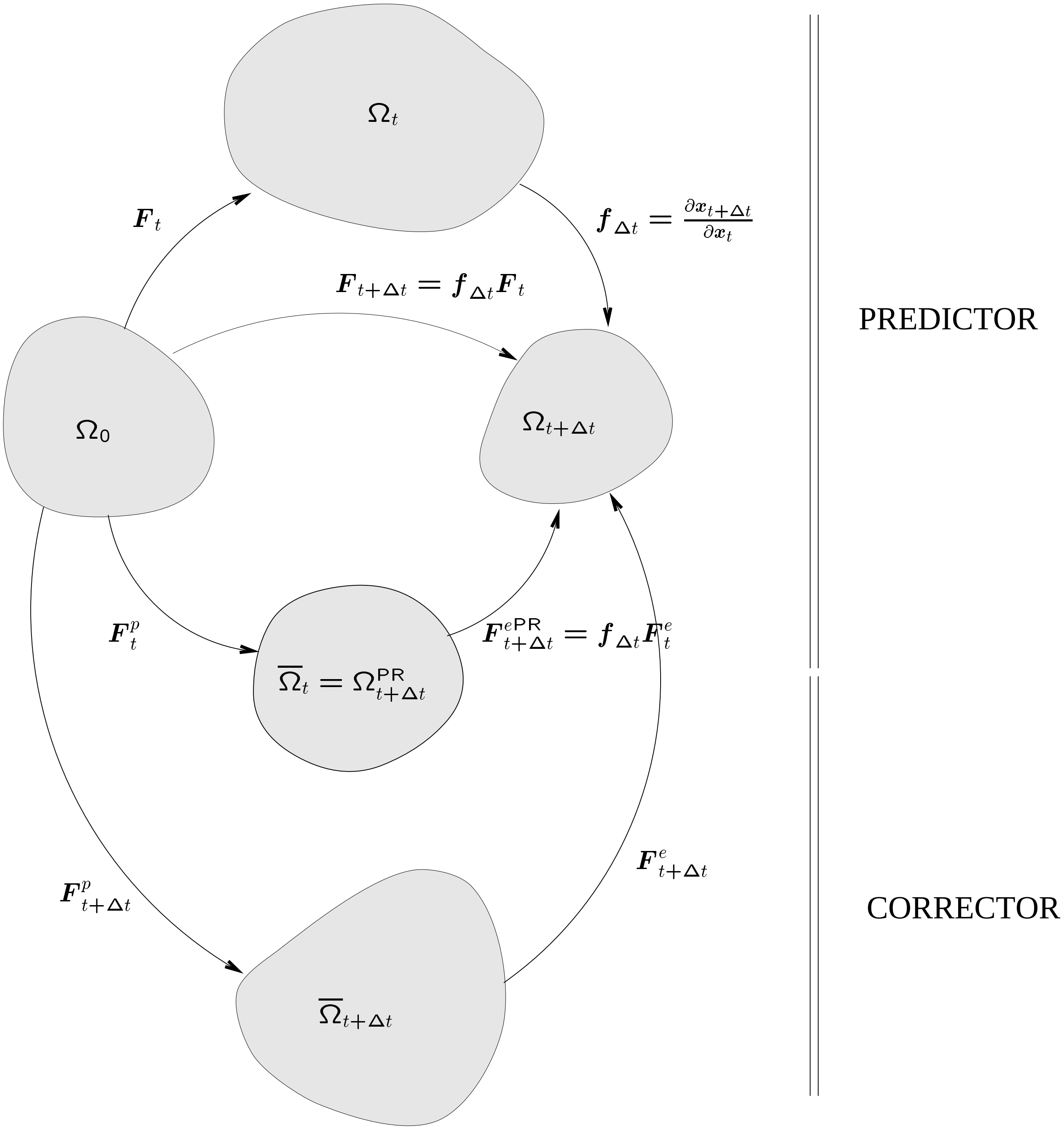
- Implementation based on the multiplicative update of the Finger strain tensor (García Garino, 1994)

- Kinematics of the element is *Total Lagrangian* and the integration of the plasticity equations is *Updated Lagrangian*

- This constitutive model has proved very efficient from a *computational cost* point of view.

INTEGRATION OF ELASTIC-PLASTIC EQUATIONS

$\boldsymbol{F}^e_t, \boldsymbol{F}^p_t, \boldsymbol{F}_t, \boldsymbol{u} : \Omega_t \rightarrow \Omega_{t+\Delta t}$ are known



ELASTIC PREDICTOR

- From the results of last step Ω_t, $\boldsymbol{b}_t^{e^{-1}}$, $\boldsymbol{F}_t$, and for a displacement increment $\Delta \boldsymbol{u}$, the predictor intermediate configuration $\Omega_{t+\Delta t}^{\mathsf{PR}}$ is evaluated:

$$\left(\boldsymbol{F}_{t+\Delta t}^{e}\right)^{\mathsf{PR}} = \boldsymbol{F}_{t+\Delta t}\boldsymbol{F}_t^{p^{-1}} = \boldsymbol{f}_{\Delta t}\boldsymbol{F}_t^{e}$$

  where:

$$\boldsymbol{f}_{\Delta t} = \boldsymbol{F}_{t+\Delta t}\boldsymbol{F}_t^{-1}$$

- Expressing the predictor deformation gradient in terms of the Finger tensor, the predictor intermediate configuration is computed via:

$$\left(\boldsymbol{b}_{t+\Delta t}^{e^{-1}}\right)^{\mathsf{PR}} = \boldsymbol{f}_{\Delta t}^{-\mathsf{T}}\boldsymbol{b}_t^{e^{-1}}\boldsymbol{f}_{\Delta t}^{-1}$$

PLASTIC CORRECTOR

The flow rule is defined in the intermediate configuration (Simo, 1985):

$$\dot{\boldsymbol{C}}^p = 2\dot{\gamma}\boldsymbol{N}$$

Integrating this equation via a Backward-Euler algorithm:

$$\boldsymbol{C}^p_{t+\Delta t} = \boldsymbol{C}^p_t + 2\gamma \boldsymbol{N}_{t+\Delta t}$$

and applying the elastic push-forward operator:

$$\Phi_* \boldsymbol{C}^p_{t+\Delta t} = \Phi_* \boldsymbol{C}^p_t + 2\gamma \boldsymbol{n}_{t+\Delta t} \qquad (1)$$

Simple calculations yield:

$$\Phi_* \boldsymbol{C}^p_{t+\Delta t} = \boldsymbol{b}^{e^{-1}}_{t+\Delta t} \qquad (2)$$

$$\Phi_* \boldsymbol{C}^p_t = \boldsymbol{b}^{e^{\mathrm{PR}-1}}_{t+\Delta t} \qquad (3)$$

Finally, substituting (2,3) in (1), the expression of the plastic corrector is obtained:

$$\boldsymbol{b}^{e^{-1}}_{t+\Delta t} = \boldsymbol{b}^{e^{\mathrm{PR}-1}}_{t+\Delta t} + 2\gamma \boldsymbol{n}_{t+\Delta t}$$

where the plastic multiplier γ is evaluated via a radial return algorithm

LOCALISATION CRITERIA

- Weak discontinuity:

  The conditions for reaching a localisation surface S with normal $\boldsymbol{n}$, are established under the following hipotheses:

  1. The field of velocities $\dot{\boldsymbol{u}}$ is smooth even when the localisation has been reached.
  2. The strain tensor may be non-smooth across the localisation surface S
  3. When localisation occurs, the loading state of the material is the same in both sides of S

- Localisation equation:

$$\boldsymbol{Q}(\boldsymbol{n}) \cdot \boldsymbol{g} = 0$$

  where:

  - $\boldsymbol{g}$ is the polarisation vector
  - $\boldsymbol{Q}(\boldsymbol{n}) = \boldsymbol{n} \cdot \boldsymbol{D}_{\text{ep}} \cdot \boldsymbol{n}$ is the acoustic tensor
  - $\boldsymbol{D}_{\text{ep}}$ is the tangent elastic-plastic modulus tensor

- The solution of the localisation equation is non-trivial if

$$\det(\boldsymbol{Q}(\boldsymbol{n})) = 0$$

CLOSED FORM EXPRESSIONS FOR LOCALISATION

- Stress tensor: incompressible uniaxial plane strain test

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & \frac{1}{2} Y \end{pmatrix}$$

- Elastic-plastic tangent moduli

$$\boldsymbol{C}_{\text{ep}} = \begin{pmatrix} \dfrac{\lambda\,\mu + \lambda\,H + \mu^2 + 2\,\mu\,H}{\mu + H} & \dfrac{\lambda\,\mu + \lambda\,H + \mu^2}{\mu + H} & 0 \\ \dfrac{\lambda\,\mu + \lambda\,H + \mu^2}{\mu + H} & \dfrac{\lambda\,\mu + \lambda\,H + \mu^2 + 2\,\mu\,H}{\mu + H} & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

- Normal vector to discontinuity surface

$$\boldsymbol{n} = (\cos(\theta), \sin(\theta))$$

- Acoustic tensor

$$Q_{11} = \frac{\lambda\,\mu \cos^2\theta + \lambda\,H \cos^2\theta + \cos^2\theta\,\mu\,H + \mu^2 + \mu\,H}{\mu + H}$$

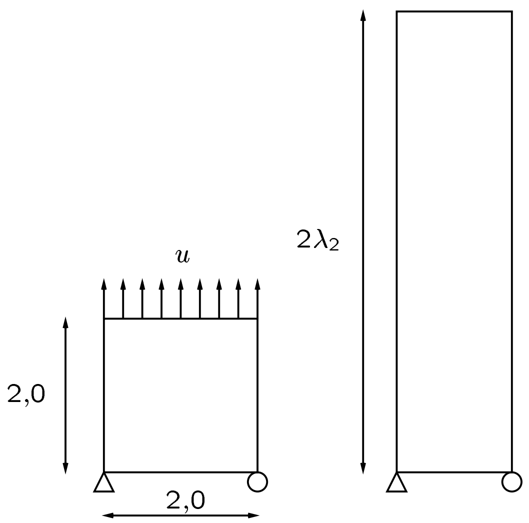
$$Q_{12} = \frac{\cos\theta \sin\theta\,(\lambda\,\mu + \lambda\,H + 2\,\mu^2 + \mu\,H)}{\mu + H}$$

$$Q_{22} = \frac{\lambda(\mu + H) \sin^2\theta + \mu\,H \sin^2\theta + \mu\,(\mu + H)}{\mu + H}$$

- Localisation condition

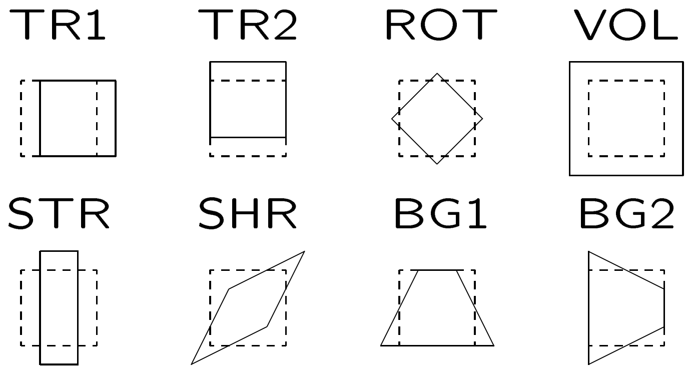
$$\det(Q) = 0 \Rightarrow H = -\frac{\mu\,(2\cos^2\theta - 1)^2\,(\mu + \lambda)}{\lambda + 2\,\mu}$$

UNCONSTRAINED SINGLE ELEMENT TEST



$$
\begin{aligned}
E &= 206{,}9\\
\nu &= 0{,}29\\
\sigma_Y &= 0{,}45 + 0{,}26\left(1 - e^{-16{,}93\varepsilon_p}\right)\\
&\quad +0{,}13\varepsilon_p\\
\lambda_2 &= 3
\end{aligned}
$$

- Eigenvectors



- Eigenvalues for the un-stressed bi-unit square element

	Q4	Q4 (R.I.)	Q4 (S.I)	$\overline{\mathrm{B}}$	Q1/E
TR1	0	0	0	0	0
TR2	0	0	0	0	0
ROT	0	0	0	0	0
VOL	$2(\mu+\lambda)$	$2(\mu+\lambda)$	$2(\mu+\lambda)$	$2(\mu+\lambda)$	$2(\mu+\lambda)$
STR	2μ	2μ	2μ	2μ	2μ
SHR	2μ	2μ	2μ	2μ	2μ
BG1	$\frac{1}{3}\lambda+\mu$	0	μ	$\frac{19\mu+\lambda}{27}$	$\frac{4}{3}\frac{\mu(\mu+\lambda)}{\lambda+2\mu}$
BG2	$\frac{1}{3}\lambda+\mu$	0	μ	$\frac{19\mu+\lambda}{27}$	$\frac{4}{3}\frac{\mu(\mu+\lambda)}{\lambda+2\mu}$

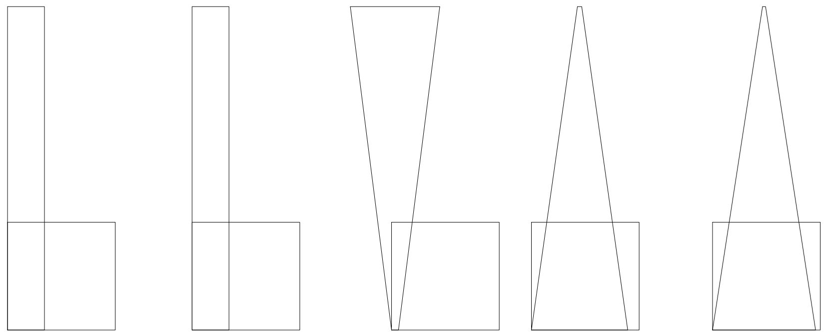
REMARKS

- The eigenvalues of the finite enhanced strain elements are equal to those of the infinitesimal Wilson-Taylor quad.

- For undistorted elements, the same eigenvalues are obtained for Q1/E4, Q1/ES4 and Q1/ET4

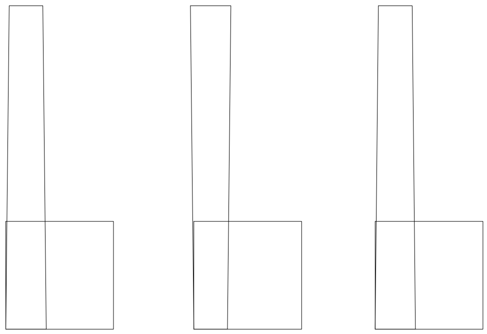
DEFORMED ELEMENTS

- Four Gauss points integration rule



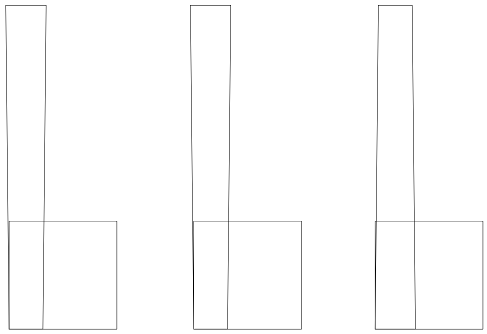
Q4 Q1/P0 Q1/E4 Q1/ES4 Q1/ET4

- Five Gauss points integration rule



Q1/E4 Q1/ES4 Q1/ET4

- Nine Gauss points integration rule



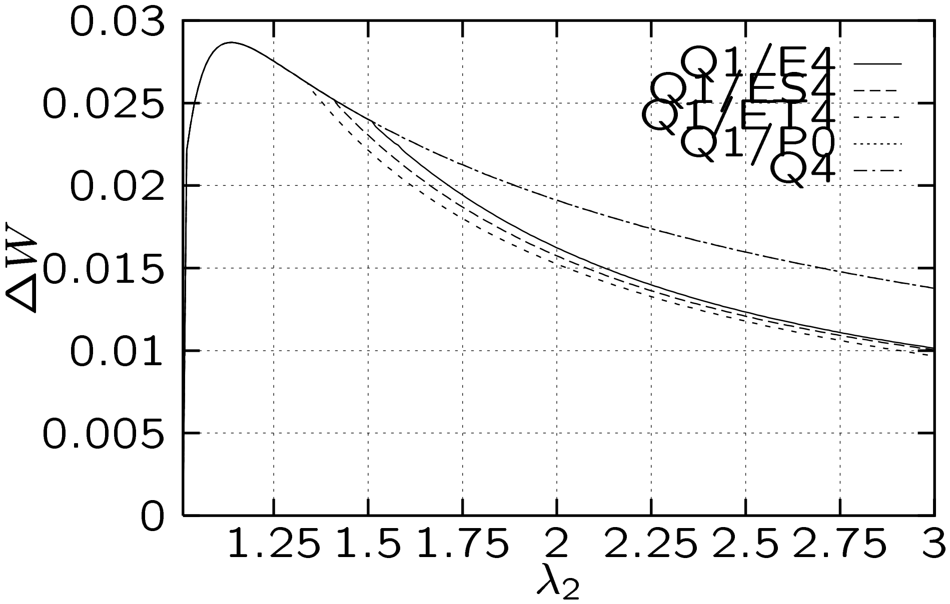
Q1/E4 Q1/ES4 Q1/ET4

REMARKS

- Severe hourglassing appears for the enhanced elements with the four points quadrature: the incompatible modes are spuriosly activated for the following values of stretching:

	Q1/E4	Q1/ES4	Q1/ET4
λ_2	1.52	1.42	1.35

This fact is clearly showed plotting the incremental deformation energy versus stretching.

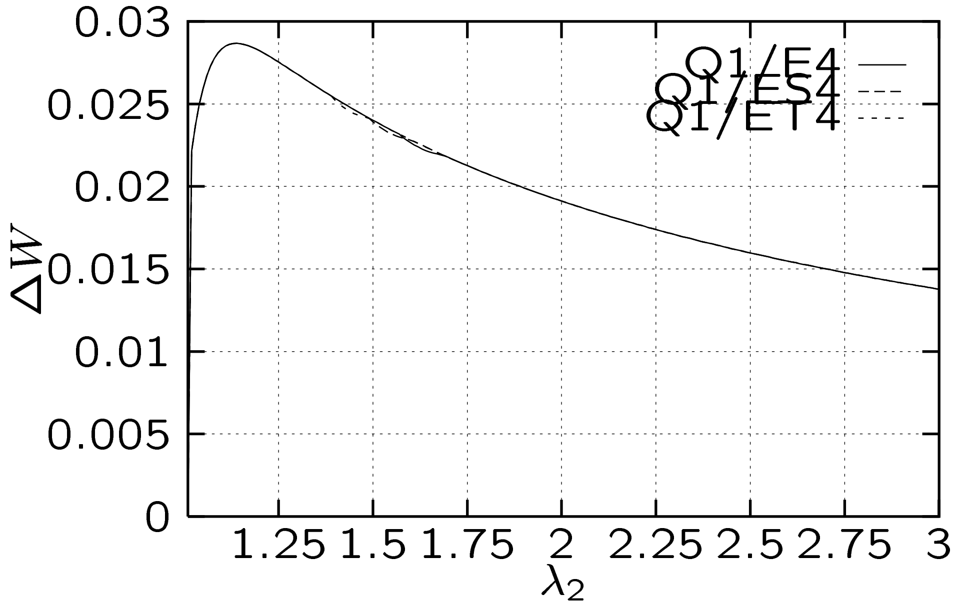


- Using 5 and 9 Gauss points, the sub-integration problem is reduced and a similar behaviour is observed for all the elements.

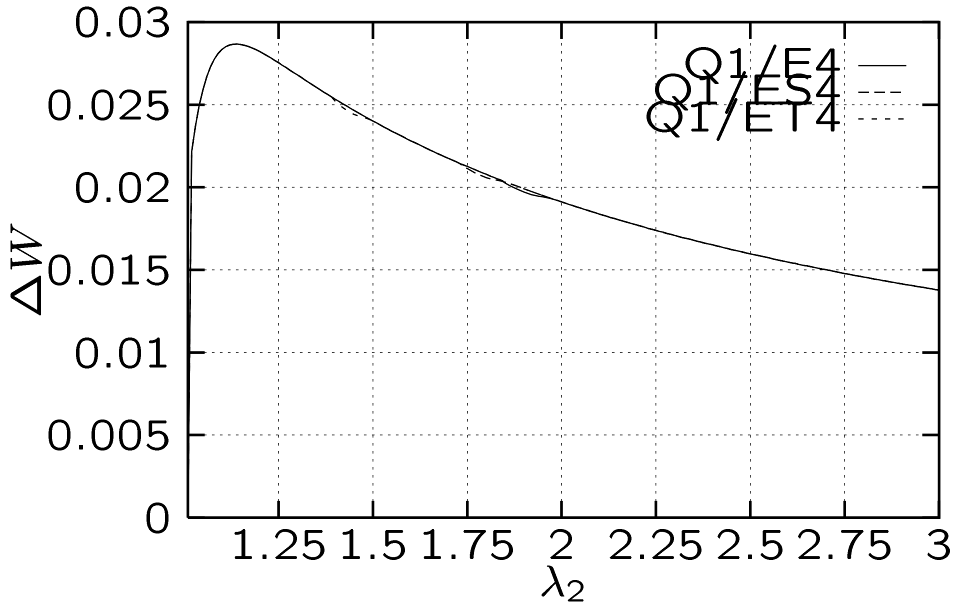
- Even for 5 and 9 Gauss points rule, a slight hourglass appear. The values of the stretching when incompatibles modes are spuriosly activated are shown in this table:

	Q1/E4	Q1/ES4	Q1/ET4
λ_2(5 G.P.)	1.57	1.49	1.39
λ_2(9 G.P.)	1.84	1.73	1.39

- Five Gauss points (Incremental deformation energy)

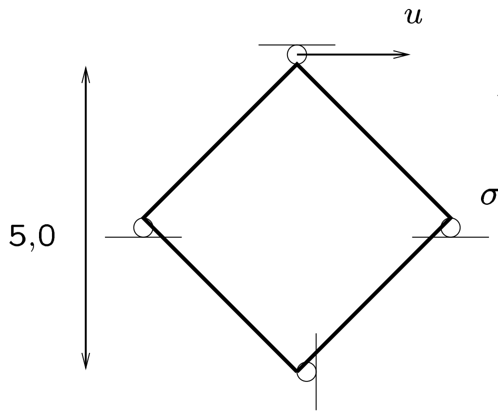


- Nine Gauss points (Incremental deformation energy)



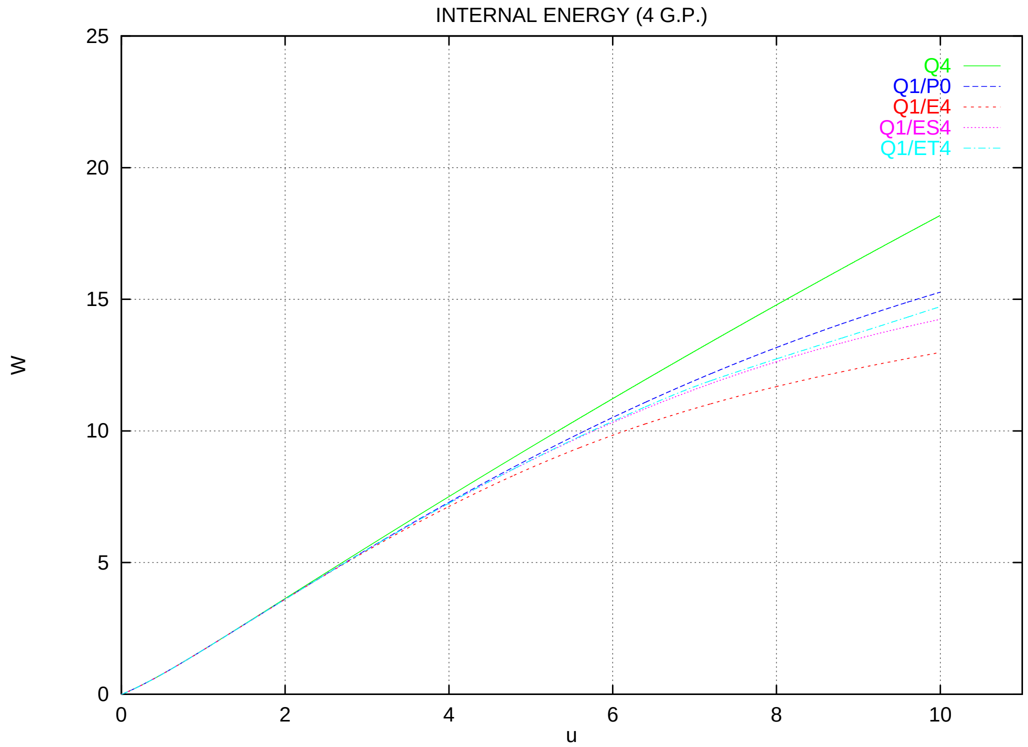
SINGLE ELEMENT LOCALISATION TEST

- Definition of the problem



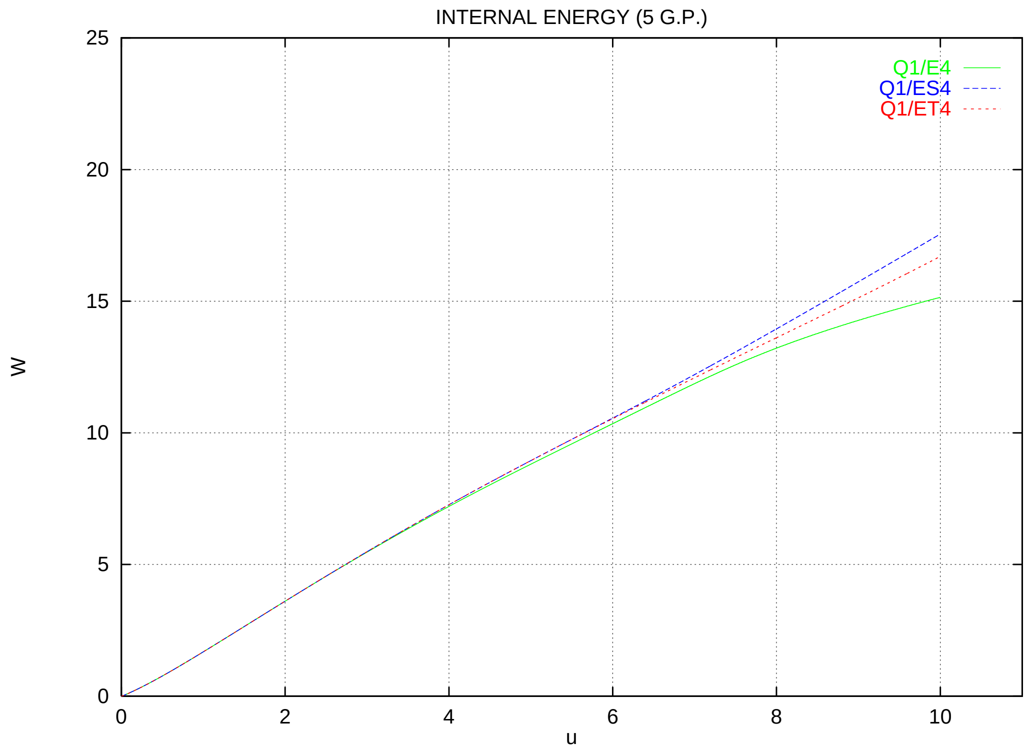
$$
\begin{aligned}
E &= 206{,}9\\
\nu &= 0{,}29\\
\sigma_Y &= 0{,}45 + 0{,}26\left(1 - e^{-16{,}93\varepsilon_p}\right)\\
&\quad -0{,}13\varepsilon_p\\
u &= 10{,}0
\end{aligned}
$$

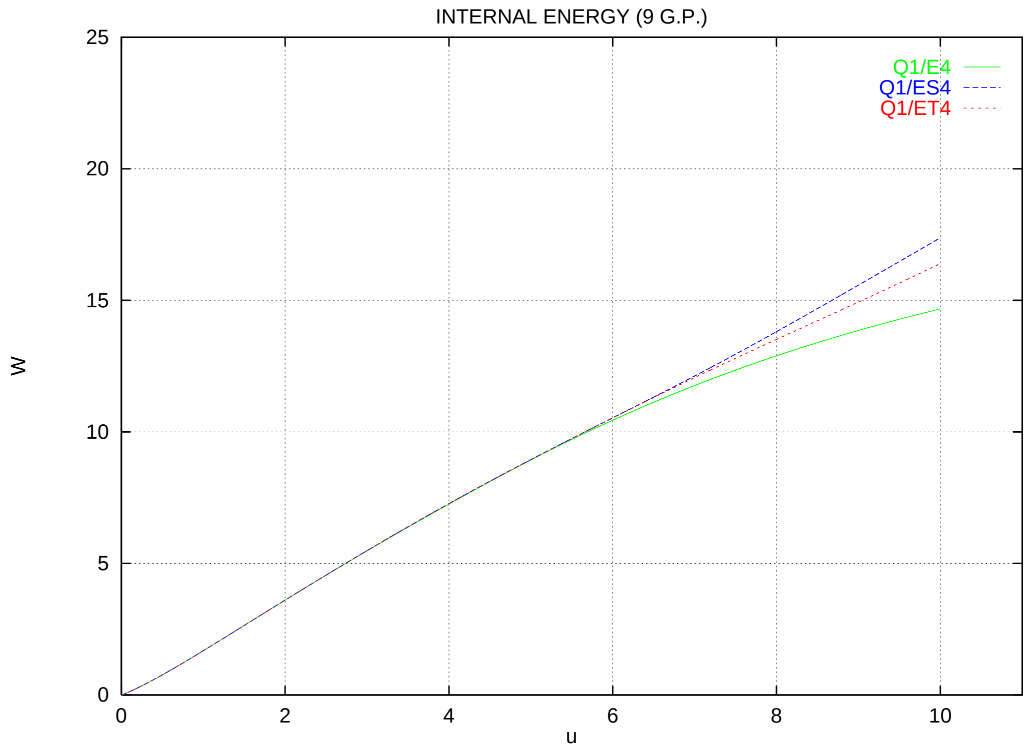
- Internal energy-displacement diagrams



SINGLE ELEMENT LOCALISATION TEST

- Internal energy-displacement diagrams



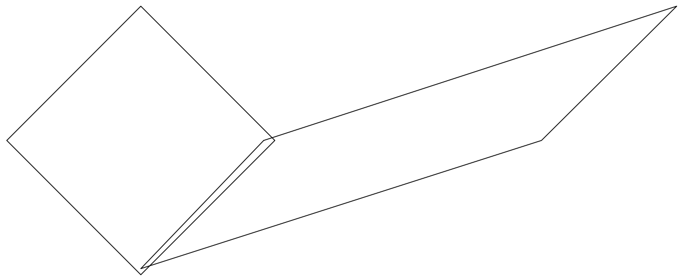


SINGLE ELEMENT LOCALISATION TEST

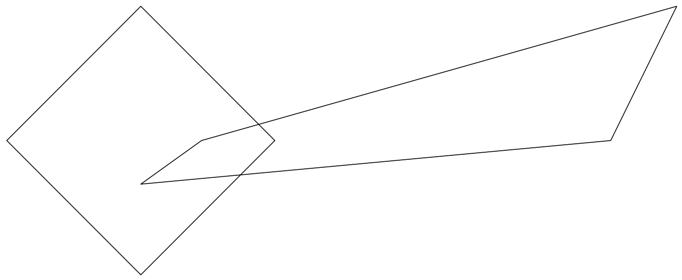
- REMARKS
  - Assuming that elastic deformations are small, a steeper softening branch in the force-displacement diagram yields a lower value of the final dissipated energy and, therefore, results in a narrower shear band.
  - This feature leads to a narrower shear band for Q1/E4 than Q1/ES4, and narrower for Q1/ES4 than Q1/ET4.
  - Attending to the quadrature rule employed, the shear band is narrower using 4 G.P than 9 G.P. and narrower for 9 G.P. than 5 G.P.

SINGLE ELEMENT LOCALISATION TEST

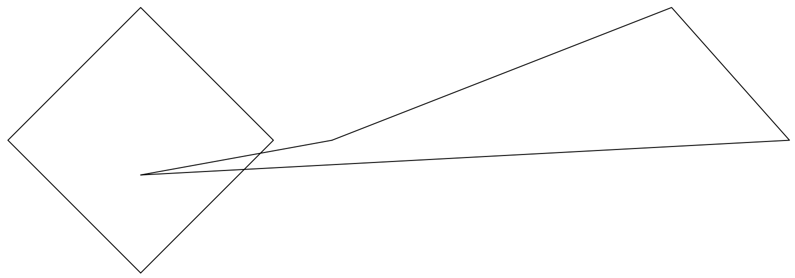
- Deformed meshes. 4 Gauss Points



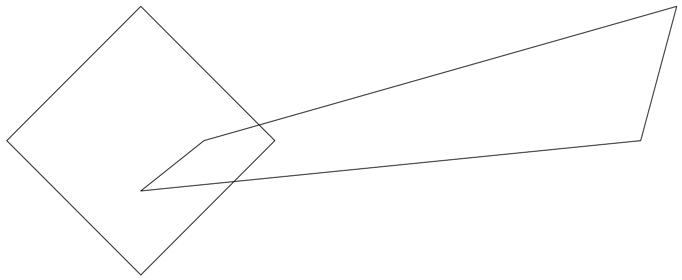
Q4



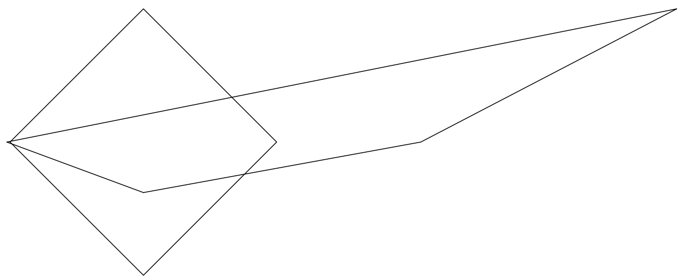
Q1/P0



Q1/E4



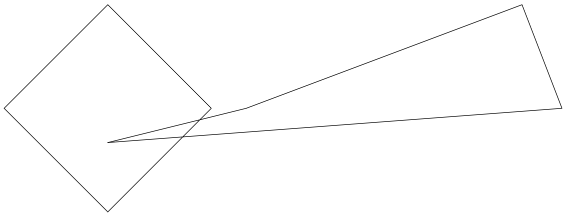
Q1/ES4



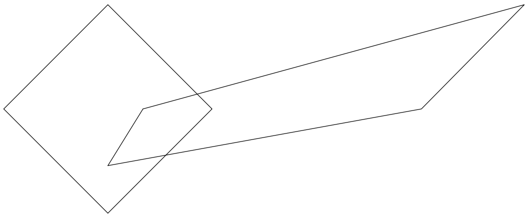
Q1/ET4

SINGLE ELEMENT LOCALISATION TEST

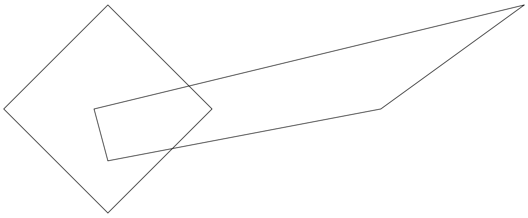
- Deformed meshes. 5 Gauss Points



Q1/E4

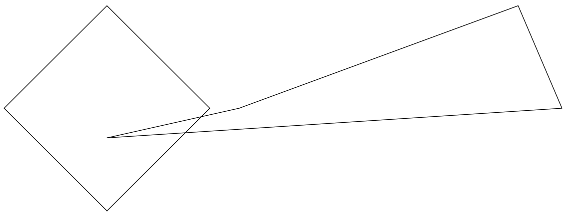


Q1/ES4

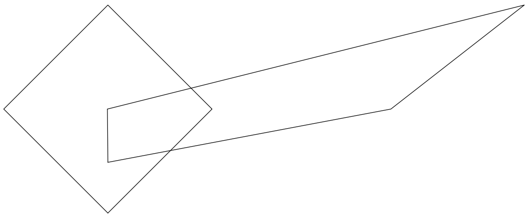


Q1/ET4

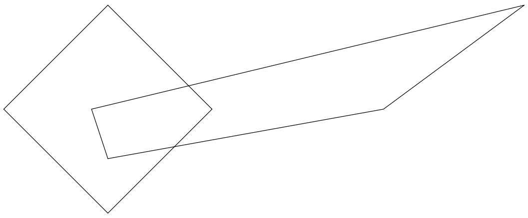
- Deformed meshes. 9 Gauss Points



Q1/E4



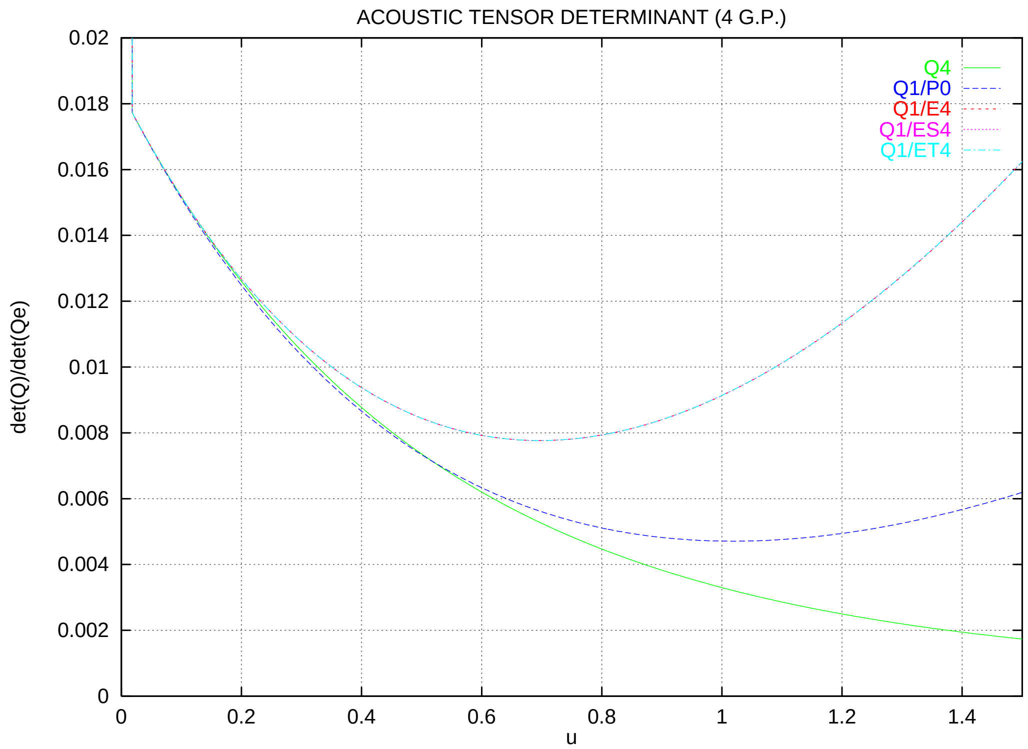
Q1/ES4

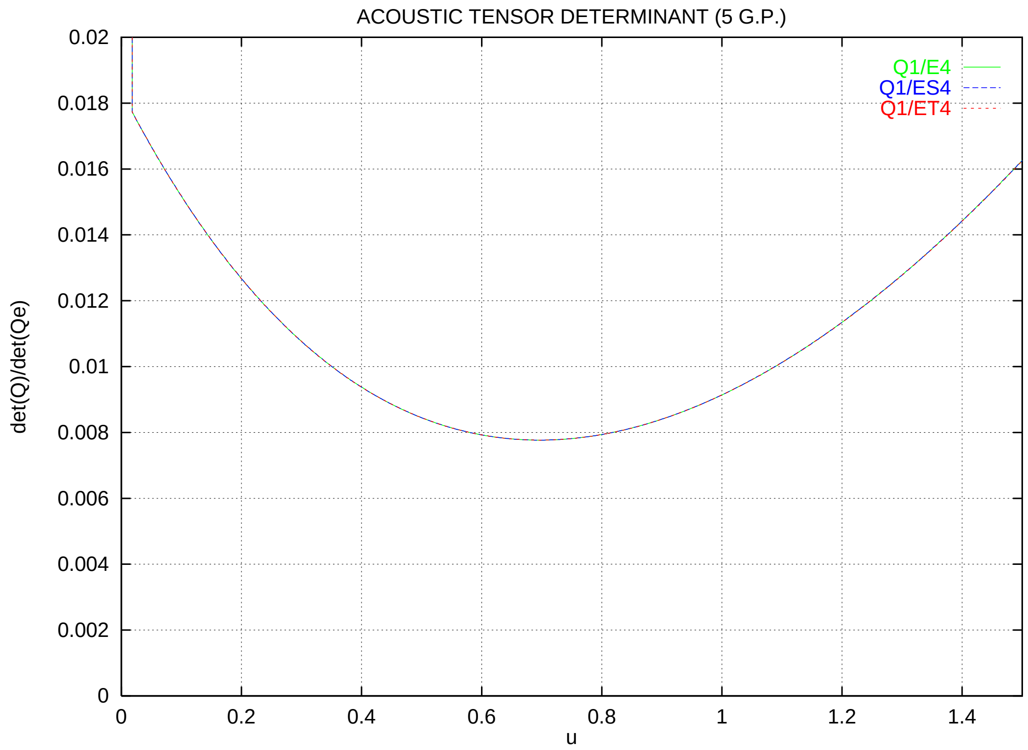


Q1/ET4

SINGLE ELEMENT LOCALISATION TEST

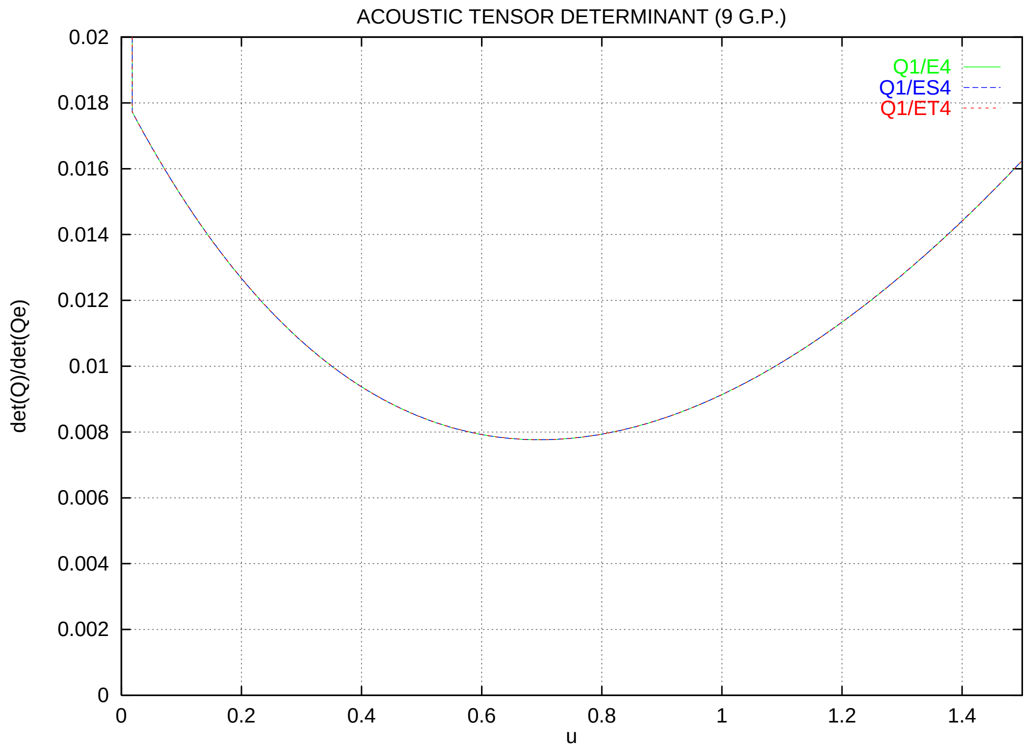
- Determinant of acoustic tensor





SINGLE ELEMENT LOCALISATION TEST

- Determinant of acoustic tensor



- Remarks
  - The shear bands are oriented along the diagonals of the square elements
  - When plastic flow appears, the determinant of the acoustic tensor always decrease for the Q4 element, and present a minimum for all the other elements.
  - The minimum is reached for the enhanced elements before than for the Q1/P0 one.
  - For the enhanced elements, the results are independent of the type of element and of the quadrature rule employed.

SINGLE ELEMENT LOCALISATION TEST

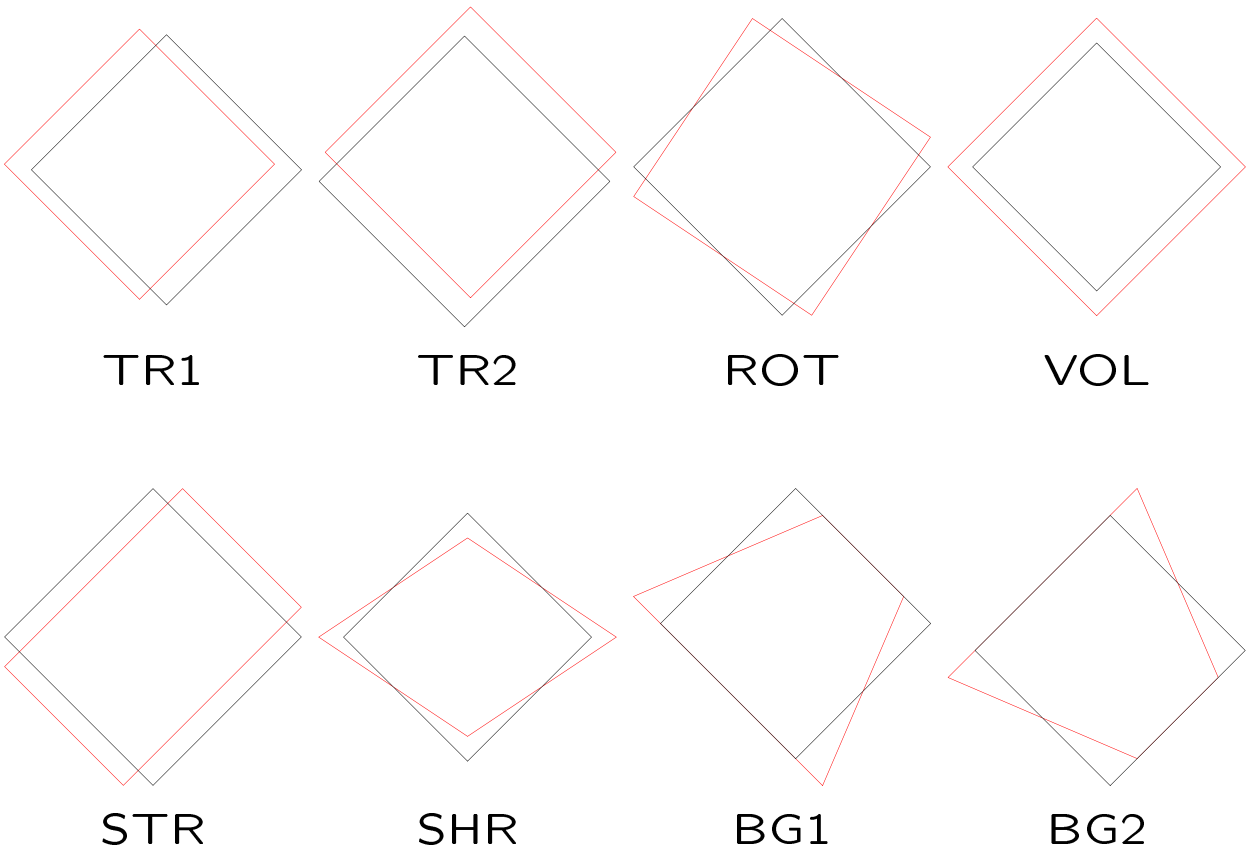
- Participation of the stiffness matrix eigenvectors during the deformation proccess:

$$\boldsymbol{d} = \sum_{i=1}^{8} p_i \boldsymbol{e}_i$$

where:

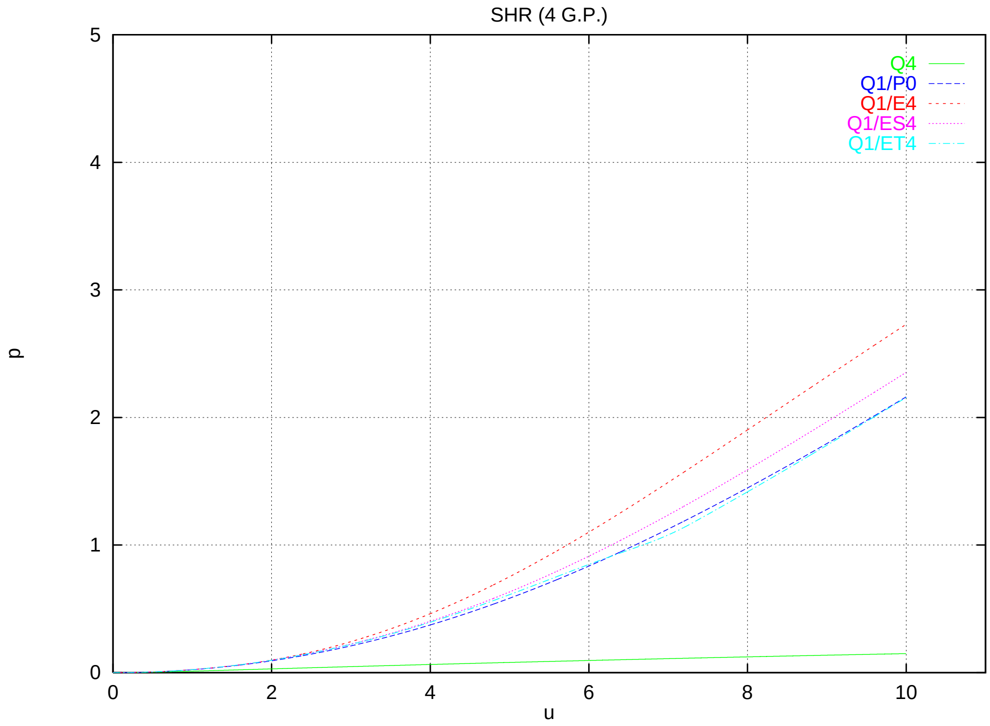
$\boldsymbol{d}$	nodal displacements vector
$\boldsymbol{e}_i$	stiffness matrix normalised eigenvectors
p_i	projection of $\boldsymbol{d}$ on $\boldsymbol{e}_i$ eigenvector : $p_i = \boldsymbol{d} \cdot \boldsymbol{e}_i$

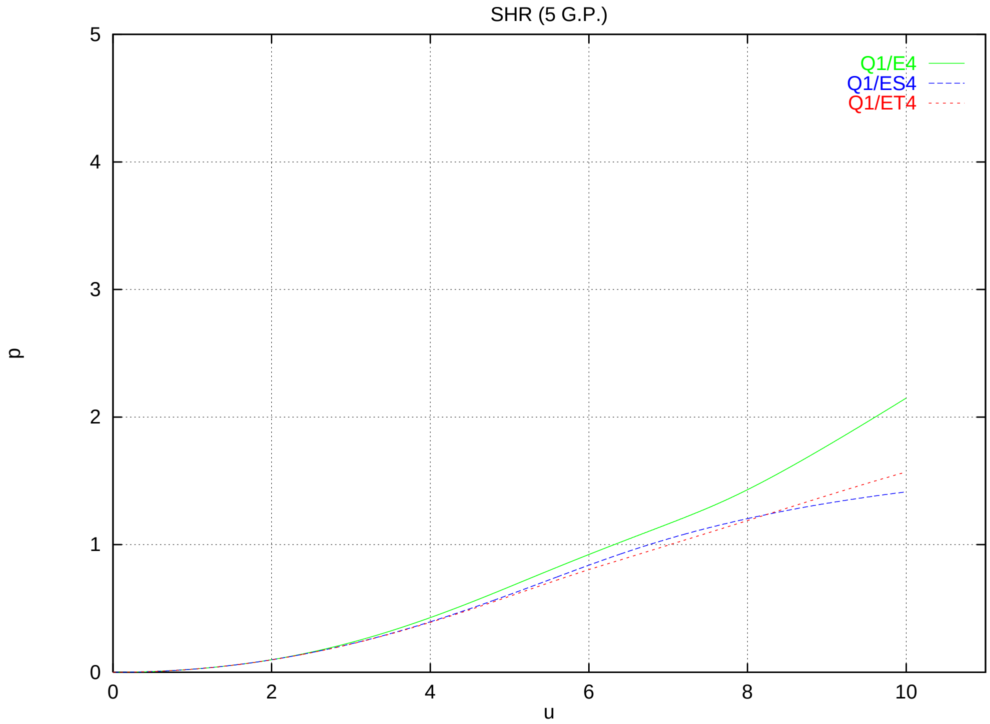
The following eigenvectors are obtained from the spectral analysis of the stiffness matrix:



SINGLE ELEMENT LOCALISATION TEST

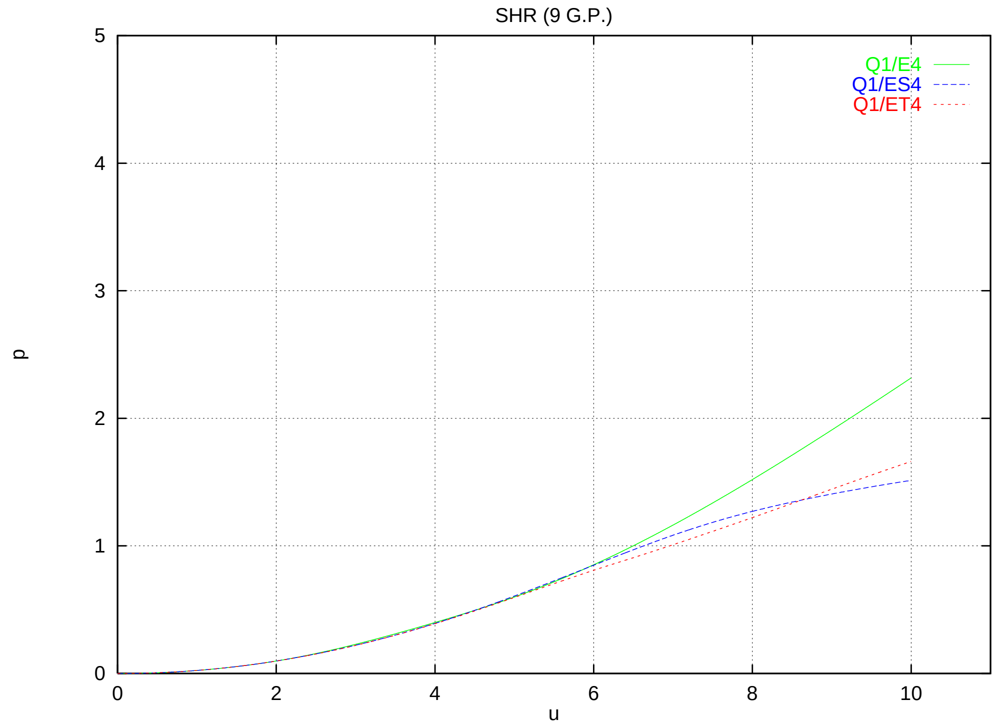
- SHR mode. Comparison of elements





SINGLE ELEMENT LOCALISATION TEST

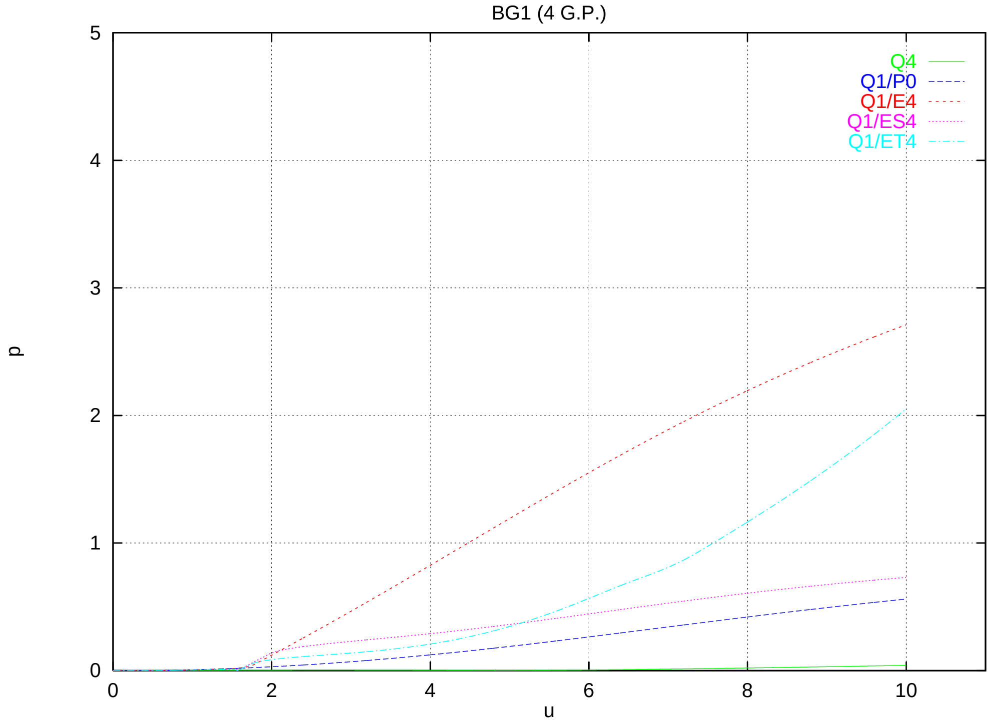
- SHR mode. Comparison of elements

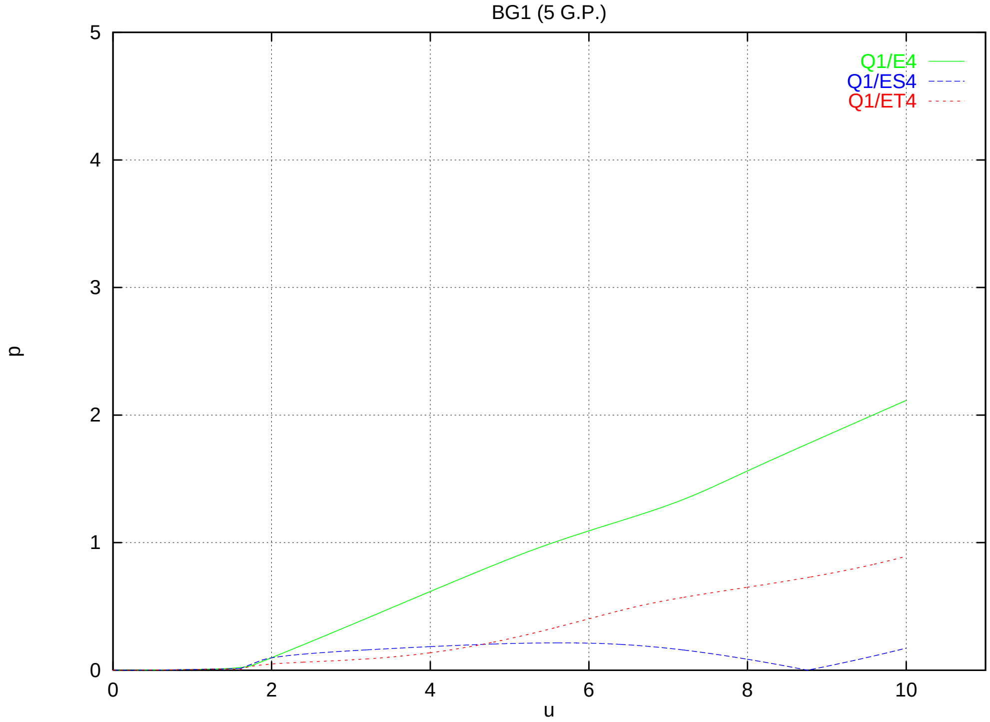


- REMARKS
  - The participation of shear mode for element Q4 is negligible if compared with enhanced elements
  - The participation of shear mode for element Q1/P0 is similar to Q1/ET4
  - The behaviour of the enhanced elements Q1/ES4 and Q1/ET4 is very similar when quadrature rules of 5 and 9 Gauss points are employed.
  - The stiffest results for shear mode are obtained with 5 Gauss points. This contribution is stiffer for 9 than for 4 Gauss points.

SINGLE ELEMENT LOCALISATION TEST

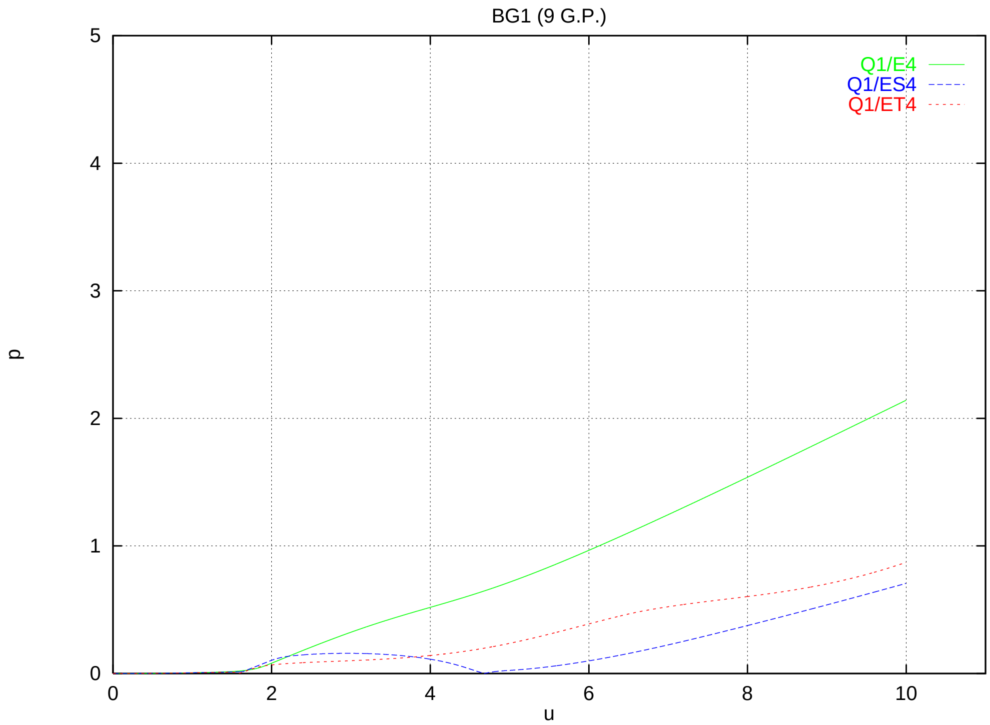
- BG1 mode. Comparison of elements





SINGLE ELEMENT LOCALISATION TEST

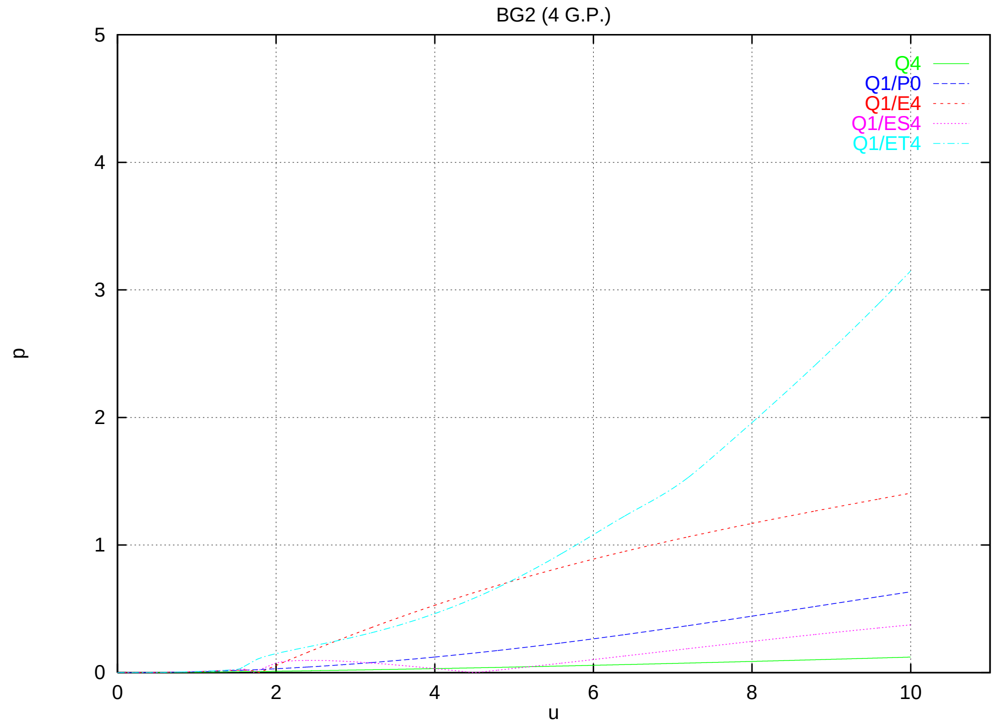
- BG1 mode. Comparison of elements

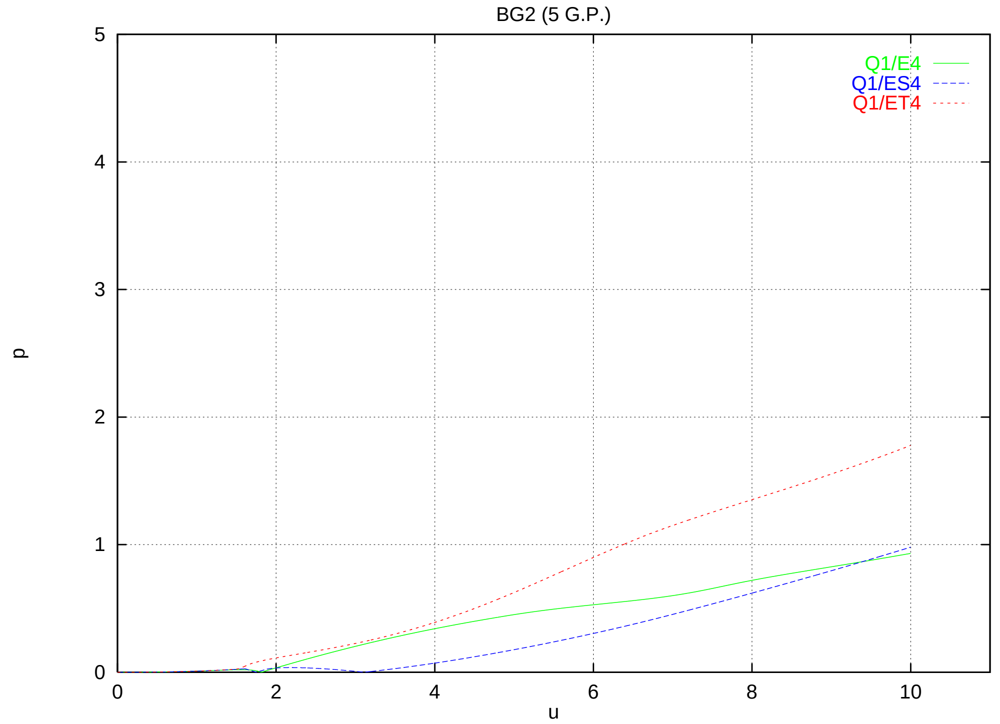


- REMARKS
  - The participation of bending 1 mode for element Q4 is negligible if compared with enhanced elements
  - The participation of this bending mode for element Q1/P0 is similar to Q1/ES4
  - The contribution for Q1/ET4 is higher than Q1/ES4 and lower than Q1/E4
  - The results obtained for Q1/E4 and Q1/ET4 with 5 Gauss points don't change using 9 Gauss points.

SINGLE ELEMENT LOCALISATION TEST

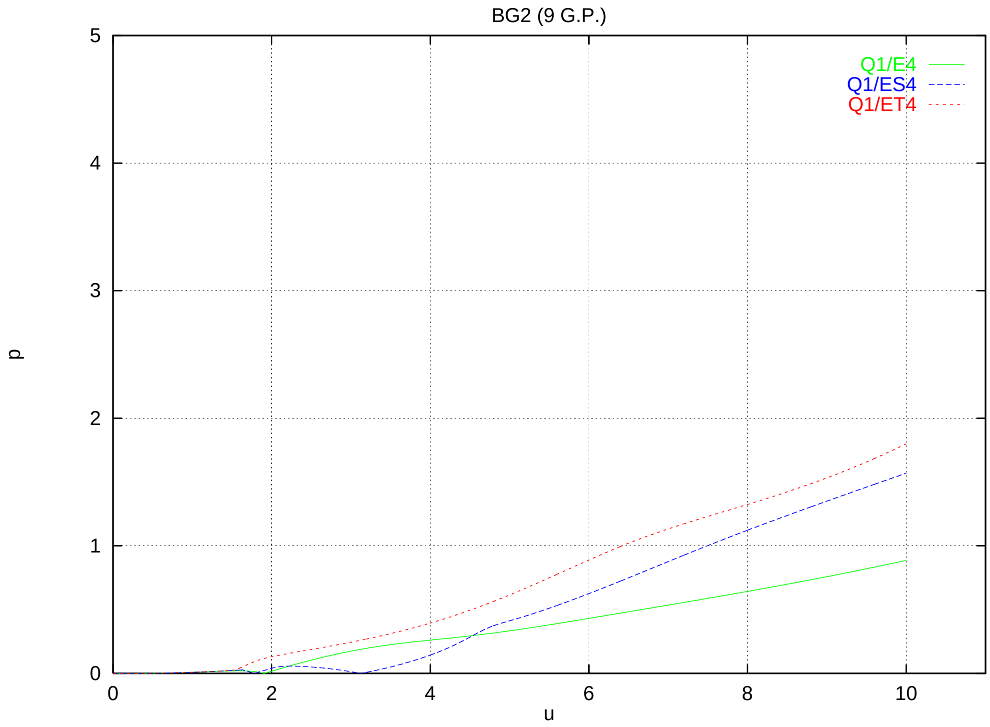
- BG2 mode. Comparison of elements





SINGLE ELEMENT LOCALISATION TEST

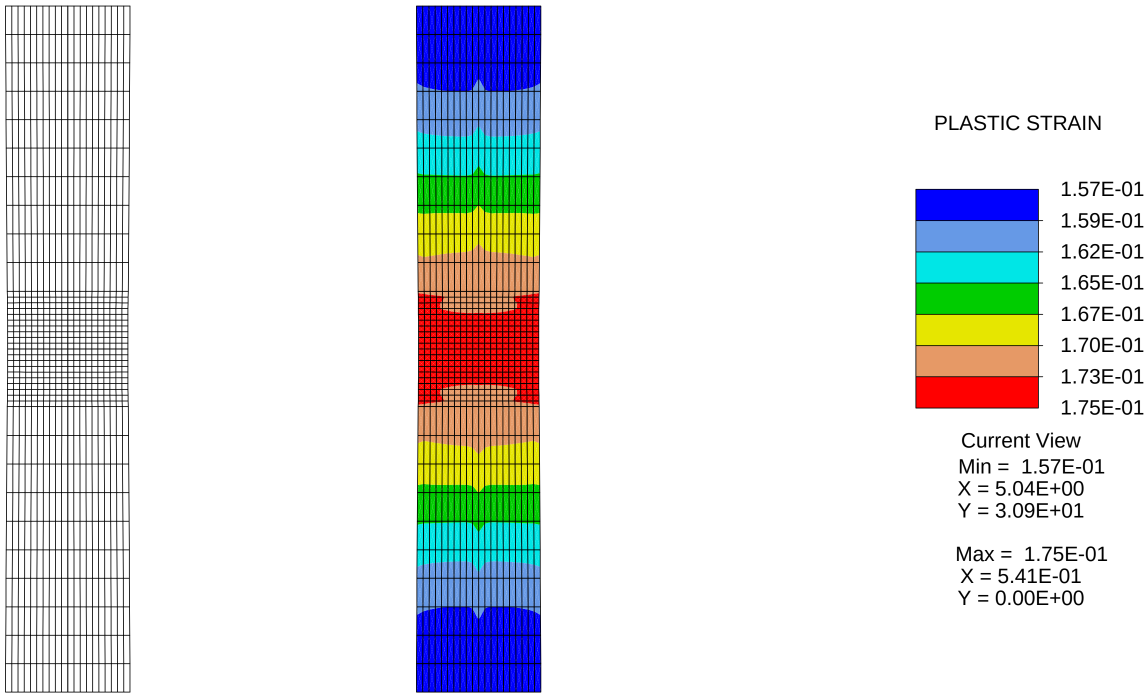
- BG2 mode. Comparison of elements



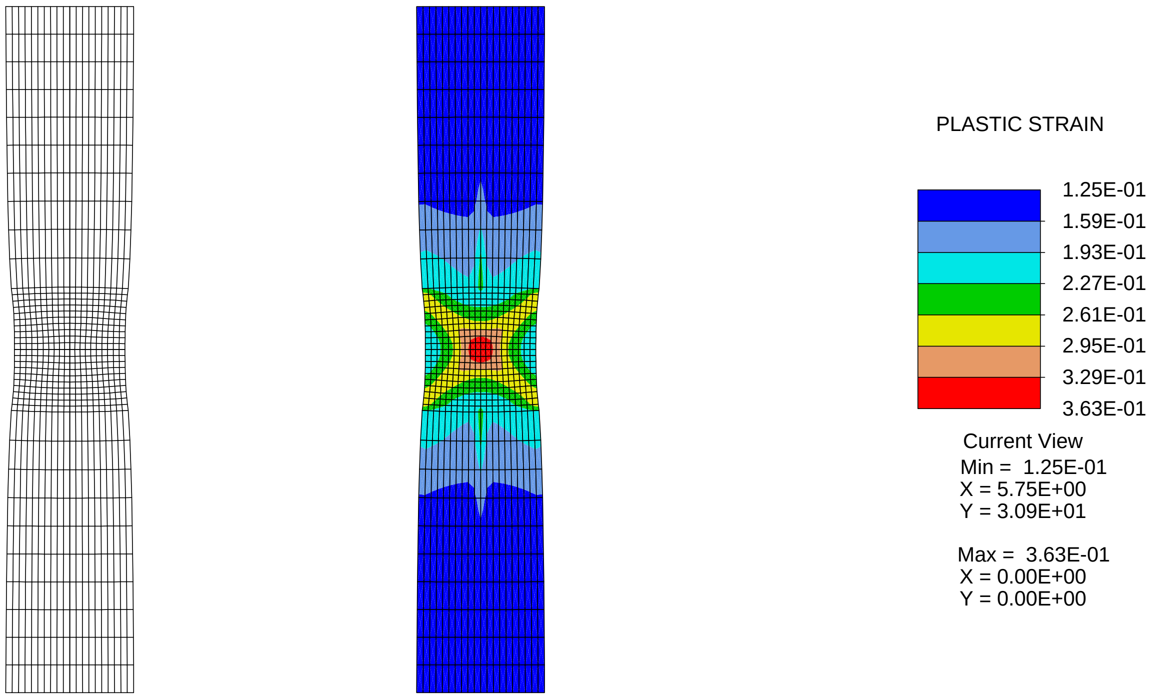
- REMARKS
  - The bending 2 mode participation is significantly higher than bending 1 mode for Q1/ET4 element and significantly lower for Q1/E4 element.
  - The contribution for Q1/ES4 increases with the order of quadrature rule. This contribution is lower than those obtained with Q1/P0 using 4 Gauss points.
  - The results obtained for Q1/E4 and Q1/ET4 with 5 Gauss points don't change using 9 Gauss points.

LOCALISATION IN A PLANE STRAIN SPECIMEN

- Q4 element

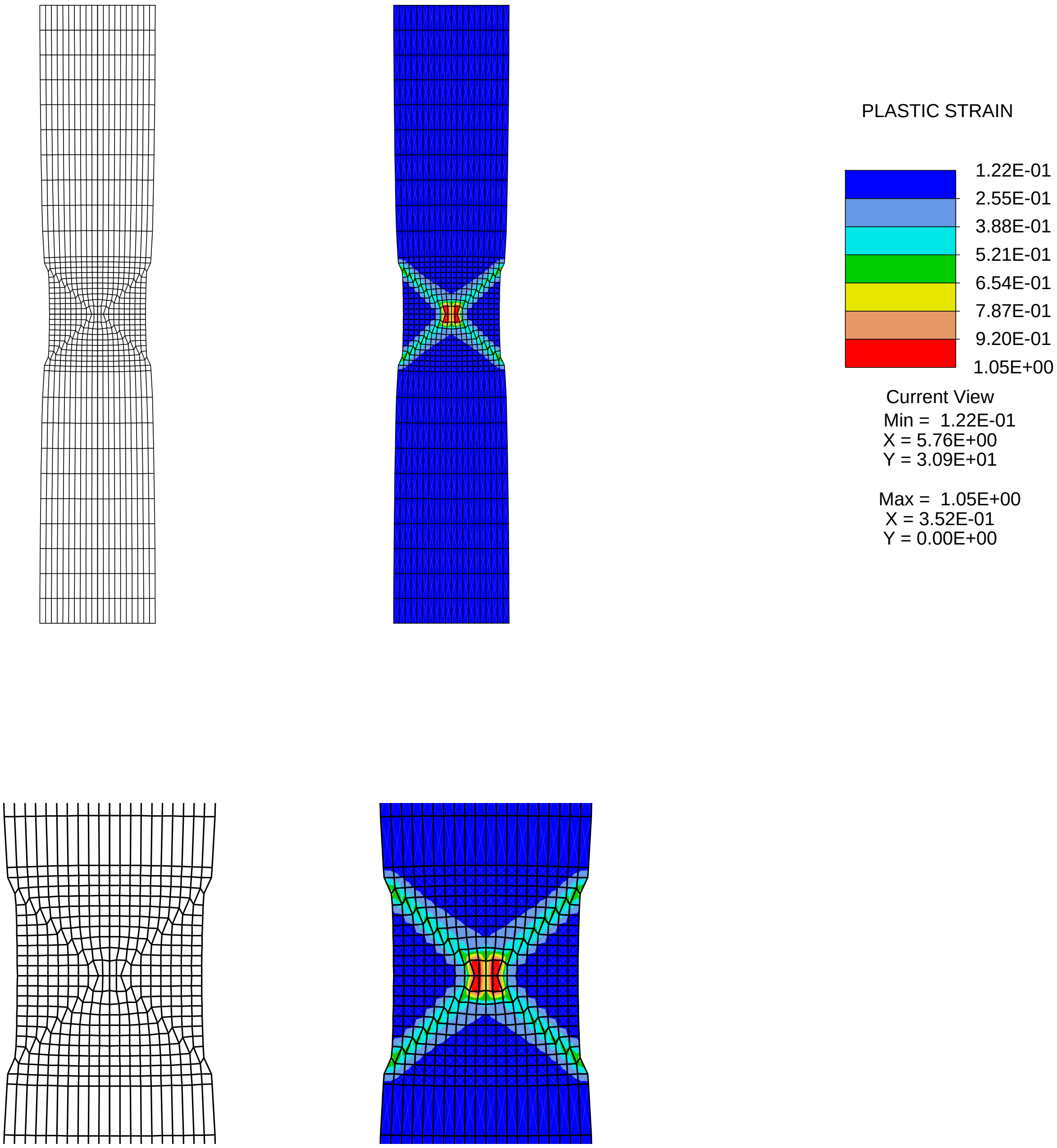


- Q1/P0 element



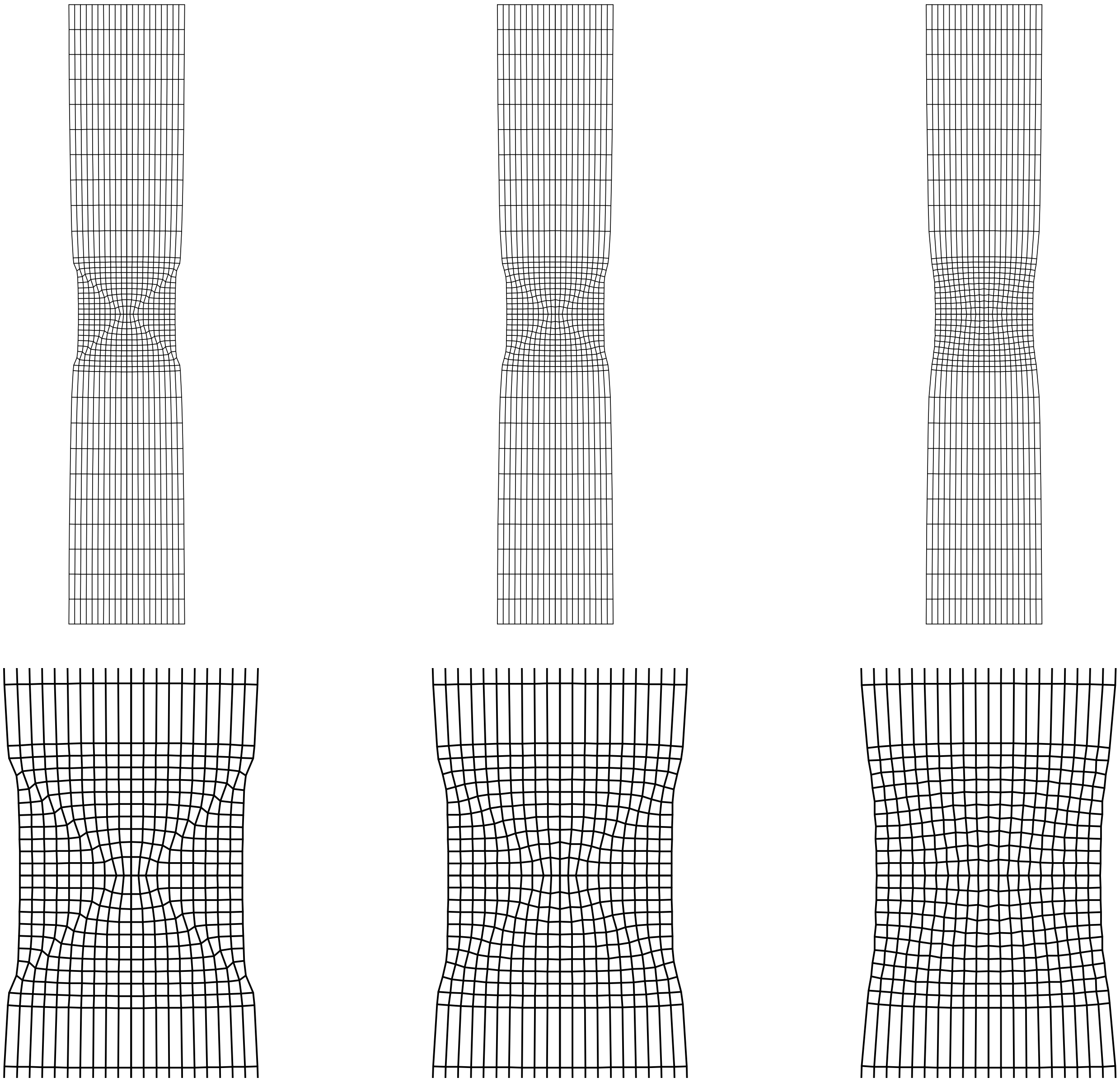
LOCALISATION IN A PLANE STRAIN SPECIMEN

- Q1/E4 element (4 Gauss points)



LOCALISATION IN A PLANE STRAIN SPECIMEN

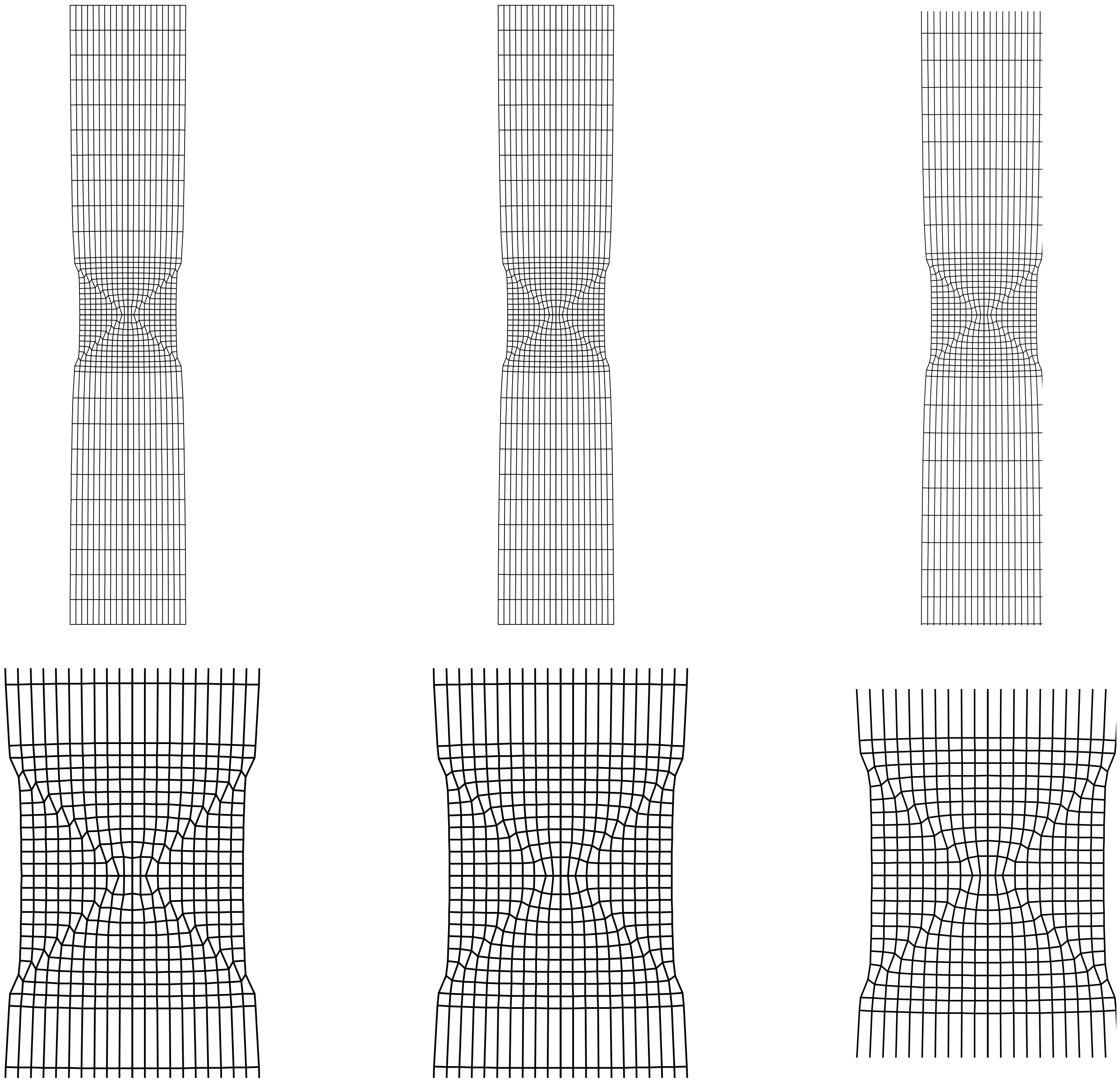
- Enhanced elements with 5 Gauss points



Q1/E4 Q1/ES4 Q1/ET4

LOCALISATION IN A PLANE STRAIN SPECIMEN

- Q1/E4 enhanced element



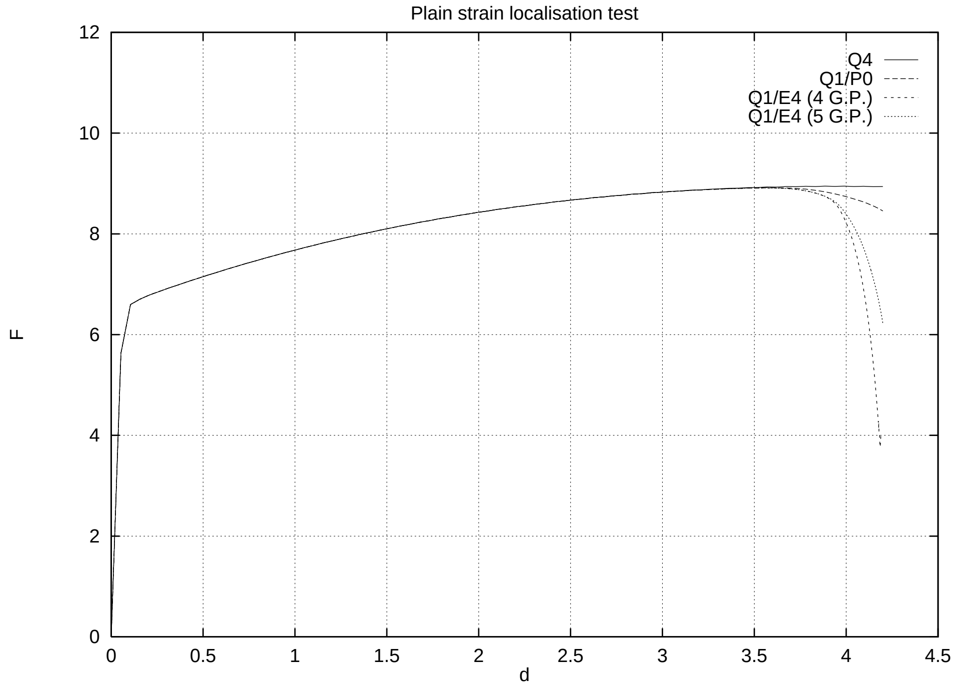
4 G.P.

5 G.P.

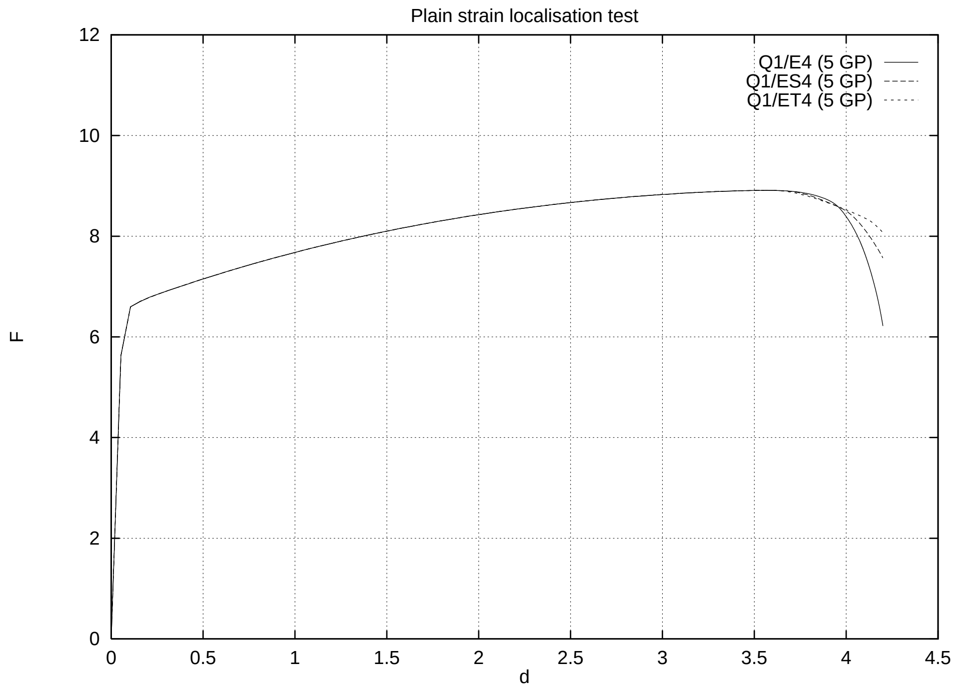
9 G.P.

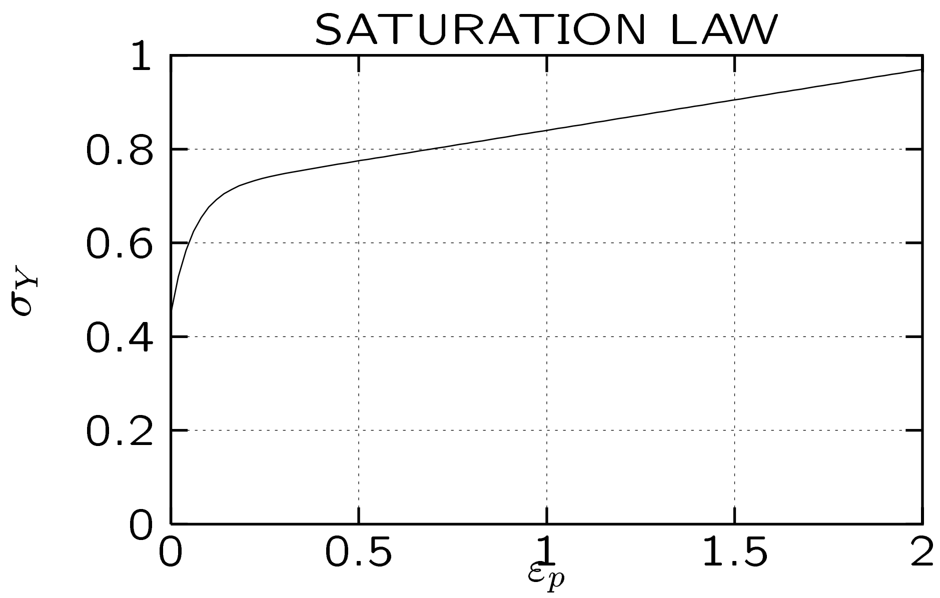
LOCALISATION IN A PLANE STRAIN SPECIMEN

- Force-displacement curve. Q1/E4 element



- Force-displacement curve. Enhanced elements




SATURATION LAW
σ_Y
ε_p
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
2